INSTITUTE OF CERTIFIED
PUBLIC ACCOUNTANTS
OF RWANDA

CPA

F1.1
BUSINESS MATHEMATICS AND QUANTITATIVE METHODS

Study Manual

2nd edition February 2020,

ICPAR
Unlimited possibilities

INSTITUTE OF CERTIFIED PUBLIC ACCOUNTANTS OF RWANDA

Foundation F1

F1.1 BUSINESS MATHEMATICS & QUANTITATIVE METHODS

2nd Edition February 2020

This Manual has been fully revised and updated in accordance with the current syllabus/ curriculum. It has been developed in consultation with experienced tutors and lecturers.

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Acknowledgement

We wish to officially recognize all parties who contributed to revising and updating this Manual, Our thanks are extended to all tutors and lecturers from various training institutions who actively provided their input toward completion of this exercise and especially the Ministry of Finance and Economic Planning (MINECOFIN) through its PFM Basket Fund which supported financially the execution of this assignment

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Stage: Foundation 1

1. Subject Title: F1.1 Business Mathematics and Quantitative Methods

Aim

The aim of this subject is to ensure that students acquire, understand and apply quantitative techniques that are used in business decision-making. They develop the ability to interpret the information obtained and present this information in a manner appropriate to a business environment.

This is an essential foundation subject for the professional accountant. It develops the mathematical and statistical competence necessary to facilitate students' progression through the Foundation and Advanced Level examinations in subjects such as Financial Accounting, Financial Reporting, Advanced Financial Reporting, Management Accounting, Managerial Finance, Strategic Corporate Finance and Strategic Performance Management.

Learning Outcomes

On successful completion of this subject students should be able to:

Demonstrate the use of financial mathematics, measures of central tendency / dispersion and indices in business.

Display information in a graphical/tabular form including frequency distributions, networks, etc.

Demonstrate the use of probability and confidence intervals in business.

Explain the concept of present value and apply discounting techniques in investment appraisal.

Apply moving averages and regression analysis in forecasting.

Syllabus

2. Introduction to Financial Mathematics

Simple and compound interest, annual percentage rate, (APR), deprecation, (straight line and reducing balance), discounting, present and future value of money and investment appraisal techniques, annuities, mortgages, amortisation, sinking funds.

Handling formulae, use of positive and negative numbers, brackets and powers, calculus.

3. Linear and quadratic equations and graphs: costs and production functions (fixed, variable and total costs, average and marginal costs):break-even analysis, revenue and profit and their interpretation

4. Sources of Data, Presentation and Use

- Sources and types of data (primary & secondary data), nature, appreciation and precautions in use.
- Role of statistics, uses and misuses of statistics in business analysis and decision making
- Presentation of data, use of bar charts, histograms, pie charts, graphs, tables, frequency distributions, histogram, frequency polygons, ogives and their use and interpretation

5. Measures of Central Tendency and Dispersion

- Averages and variations for grouped and ungrouped data
- Measures of location – mean, median, mode, geometric mean, harmonic mean, percentiles, quartiles.
- Measures of dispersion – range, variance, standard deviation, co-efficient of variation

6. Probability and Probability Distributions

- Meaning of probability, nature of probability distributions, discrete and
- Continuous random variables, expected values. Standard Normal Distribution, confidence intervals, z-score, T-Chi square and associated diagrams
- Use and application of probability distributions.
- Binomial probability distribution and its application in business
- Analysis of binominal populations (the probability of success p, and failure q)
- The Poisson Probability Distribution, the Poisson population and application of Poisson distribution in analyzing of Poisson events.

7. Sampling and Sampling Theory (The role of sampling as compared to population census)

- Probability Sampling Methods – Simple random, stratified, cluster, Systematic sampling
- Interval estimation for large and small samples; confidence levels, standard error; estimate of sample size.
- Hypothesis testing – Null and Alternative hypothesis; description of Type I and Type II errors. Non Probability sampling methods = quota sampling and snowball sampling.

8. Regression and Correlation Analysis

- Simple Linear Regression, scatter graphs, least squares method.
- Co-efficient of determination, correlation co-efficient, rank and product moment correlation.
- Use of linear regression equation in forecasting.

9. Time Series Analysis

- Factors influencing time series – trend, seasonal, cyclical, irregular variations.
- Smoothing time series by means of moving averages.
- Use of time series in forecasting

10. Indices: Use And Construction

- Simple, aggregate, Laspeyres, Paasche, chain indices.
- Change of base period, weighting.
- Construction, use and interpretation of indices.

11. Network Analysis

- Activity identification, Relationship between various elements, construction of simple networks.
- Analysis of networks by deriving the critical and non-critical activities.
- Derivation and definition of the critical path.

12. Linear Programming

- Simple Linear Programming and simplex
- Transportation
- Assignments

13. Decision Theory

- Minimax, Maximum, Maximax
- Decision Trees
- Game Theory

CHAP 1.

LINEAR ALGEBRA AND CALCULUS

1.1 Study objectives

By the end of this chapter, you should be able to:

- solve linear equations;
- solve quadratic equations by both factorization and completing the square methods;
- solve simultaneous equations involving two or three variables; and
- formulate and solve simultaneous equations from given word problems.

1.2. Linear equations

1.1.1. Linear equation in one variable

Linear equations are generally expressed in the form; ax + b = 0, where values a and b are constants. For example, 3x + 6 = 0, x - 2=0, 6 - 3x = 0. Also, linear equations can be in the form $\frac{2x-1}{3} = \frac{x+102x-1}{4\quad 3} = \frac{x+10}{4}$ which can be expressed in the general form.

The equation is solved by finding the unique value of x that satisfies the equation.

Example 1.1

Given the equation 2x-8 =0.

Required:

Solve the given equation.

Solution:
Given 2x - 8=0

2x- 8 + 8 = 0 + 8 (adding 8 to both sides)

Then. 2x = 8
SO, $\frac{2x}{3} = \frac{82x}{2\ 3} = \frac{8}{2}$(dividing through by 2)

x = 4

Example 1. 2

From the 4x+3=33

Required:

Find the values of x which satisfies the equation.

Solution:

4x + 3 = 33

Then, 4x + 3 - 3 = 33 - 3 (subtracting 3 from both sides)

4x = 30

And, $\frac{4x}{4} = \frac{304x}{4\ 4} = \frac{30}{4}$

x = 7.5

Example 1. 3

The heights of tomato plants exhibited on a farmer's trade show satisfied the equation $\frac{x+1}{1+2x} = \frac{1\ x+1}{31+2x} = \frac{1}{3}$

Required:

Find the value of x.

Solution:

$$\frac{2x+1}{1+2x} = \frac{1}{3}$$

Then 3(2x-1) =1(1+2x) (by cross multiplication')

6x-3 = 1+2x

And 6x-2x = 1+3 (subtracting 2x and 3 from both sides)

4x=4

$$\frac{4x}{x} = \frac{4}{4}$$

x= 1

1.1.2. Simultaneous linear equations (in two variables)

These equations take on the general form:

ax + by = c

dx + ey = f,

where a, b, c, d, e and f are constants.

The simultaneous linear equations can be solved by finding the unique values of x and y that satisfy both equations. There are different methods of solving simultaneous linear equations. They include elimination and substitution, matrix method and graphical method.

Example 1. 4

From the simultaneous equations below:

7x-3y = 41

3x-y = 17

Required:

Solve for x and y by matrix method.

Solution:

Given 7x - 3y = 41

3x-y = 17

Forming the matrix equation:

$$\begin{pmatrix} 7 & -3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 41 \\ 17 \end{pmatrix}$$

The matrix $\begin{pmatrix} 7 & -3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 7 & -3 \\ 3 & -1 \end{pmatrix}$ is a coefficient matrix of the simultaneous linear equations.

First find the inverse of the coefficient matrix

$$\frac{1}{2}\begin{pmatrix} 1 & -3 \\ 3 & -7 \end{pmatrix}$$

Pre-multiplying both sides by the inverse of the coefficient

$$\frac{1}{2}\begin{pmatrix} 1 & -3 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 42 \\ 17 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Then x = 5

And y = -2

Example 1. 5

Given the. Simultaneous equations:

y- 3x = 5

2y + 3x = 28

Required:

Solve simultaneous equations by the method of elimination and substitution.

Solution:

y-3x = 5..(1)

2y + 3x = 28...(2)

(1) + (2) - Elimination y

3y + 0 = 33

$$\frac{3y}{3} = \frac{6}{3}$$

∴∴ y = 11

Substituting for y in equation (1);

11 - 3x = 5

-3x = -6

$$\frac{-3x}{-3} = \frac{-6}{-3}$$

∴ ∴ x = 2

Hence, x =2 and y = 11

Example 1. 6

Given the following simultaneous equations:

X + 3y = -1

-3x + 6y = 8

Required:

Solve the given simultaneous equations by the substitution method.

Solution:

X + 3y = -1...(1)

-3x + 6y = 8..(2)

From equation (1)

X = -1 -3y ....(3)

Substituting for x in (2):

-3(-1 -3y) + 6y = 8

3 + 9y + 6y = 8

15y = 5

$$\frac{15y}{16} = \frac{5}{16}$$

∴∴ y $= \frac{11}{33}$

And using (3):

$x = -1-3(\frac{11}{33})$

$\therefore x = -2$

Hence, $x = -2$ and $y = \frac{11}{33}$

Example 1.7

A rally driver in a competition is expected to cover a distance of 285 km in five hours. Part of the race track is murram while the rest is under repair. The drive is to cover the section under murram at 60km/h and the section under repair at 45 km/h.

Required:

How long is the murram section?

Solution:

Let x and y be the distance covered along the murram and the section under repair respectively:

Considering the distance covered: $x + y = 285$……………………………………….(i)

Considering the time taken: $\frac{x\ x}{6060} + \frac{y\ y}{4545} = 5$

Multiplying all through by 60x45:

$60 \times 45(\frac{x\ x}{6060}) + 60 \times 45(\frac{y\ y}{4545}) = 60 \times 45 \times 5$

$45x + 60y = 60 \times 45 \times 5$

Dividing all through by 15:

$3x + 4y = 900$...(ii)

4(i) - (ii) - eliminating y

$4x + 4y = 980$

Subtracting $\underline{3x + 4v = 900}$

$x + 0 = 80$

$\therefore$ x = 80 km, the length of the murram section of the track.

1.1.3. Simultaneous linear equations (in three variables)

These equations take on the general form:

$ax + by + cz = d$

$ex + fy + gz = h$

$ix + jy + kz = l$

where a, b, c, d, e, f, g, h, i, j, k, l are constants.

A typical example here is:

3x + 2y - z = 5

2x - 2y +3z =-3

4x + y -3y = 2

The equations can be solved using several different methods to find the unique values of x, y and z that satisfy all the three equations.

Procedure for solving linear equations with three variables:

The following can be useful and should be implemented so as to solve such problems with ease:

- Beginning with any two of the given equations, eliminate one of the variables.

This gives rise to an equation with two variables.

- Using the remaining equation (the one not used in i above) and any of those used above, eliminate the same variable as in the first step. This will generate another equation with two variables.
- The two equations obtained above are then solved simultaneously by any of the methods already covered above.
- Substituting in any of the original equations, the variable eliminated earlier will then be found.

Example 1. 8

Solve the simultaneous equations below:

3x - y + z = 5

2x + 2y +3z = 4

x + 3y-z =11

Solution:

Given that:

3x-y + z = 5.. (1)

2x + 2y +3z = 4.. (2)

x + 3y-z=11.. (3)

From, (I) + (3): 4x + 2y = 16.. (4)

3(3) + (2):5x + 11y = 37.. (5)

Equations **(4)** and (5) are solved as simultaneous equations in two variables:

By elimination method and eliminating x first:

5**(4)** - 4(5); 34y =68, that is, y = 2

Substituting for y in equation **(4)** gives:

4x +2(2) = 16

4x = 12, i.e. x = 3

∴ ∴ x = 2, y = 3

Note: Any of the original equations can be used to check the accuracy of the

Example 1. 9

In a car assembly plant the production process goes through three stages preparation, progress and the finishing stages each requiring the use of fundamental components A, B and C as follows:

Preparation stage requires 300 components of type A, 400 components of type B, and 200 components of type C; the progress stage requires 200 components of type A, 300 of type B and 100 of type C and the finishing stage requires 600 components of type A, 1,000 of type B, and 1,000 of type C. Preparation stage costs Frw 460,000, progress stage costs Frw 300,000 and finishing stage costs Frw 1,440,000.

Required:

Calculate the cost of each of the components A, B and C.

Solution:

Formulation of the problem:

Cost for preparation stage - 300a +400b+ 200c = 460,000

↔ 3a + 4b + 2c = 4,600 .... (j)

200a + 300b + 100c = 300,000

↔ 2a + 3b + c = 3,000 .... (ii)

600a +1,000b + 1,000c = 144,000

↔ 3a+ 5b + 5b = 7,200 .... (iii)

3a + 4b + 2c = 4,600 .... (i)

4a + 6b + 2c = 6,000 .... (ii)x2

Subtracting: -a - 2b =-1,400

↔ a + 2b = 1,400 .... (iv)

10a + 15b + 5c = 15,000 .... (ii x5

3a+ 5b + 5c = 7,200 .... (ii)

Subtracting: 7a + 10b = 7,800 .... (v)

7a + 14b = 9,800 .... (iv;x7

7a + 10b = 7,800 .... W(v)

Subtracting: 4b-= 2.000

$$\frac{4b}{4} = \frac{2000}{4}$$

b = 500

Substituting for b in (iv); a + 2(500) = 1,400

a = 1,400-1,000

∴a = 400

Substituting for a and b in (ii)

2(400) + 3(500) + c = 3,000

c = 3,000 -2,300

∴c = 700

Hence, a = Frw 400, b = Frw 500 and c = Frw 700.

1.2. Quadratic equations

A quadratic equation in one variable has a general form **ax^2 +bx + c =0, w**here a, b and c are constants that represent numerical values and x is a variable.

Remember, a≠0 and the highest power of the variable is always two (2).

Note: To be able to handle quadratic equations with ease it is important to pay attention to the following:

- The general form of a quadratic equation.
- The relationship between the quadratic equation and their plotted curves.
- Both the algebraic and the graphical methods of solving them.

Examples of quadratic equations:

i. $2x^2 + 3x - 8 = 0$ (a = 2, b = 3, c = -8).

ii. $6-7x^2 = 0$(a = -7, b = 0, c =-6).

iii. $0.5x -1 2x^2 = 0$ (a = -1.2, b = 0.5, c = 0).

iv. $\frac{xx}{44}-\frac{-3-3}{x\ x}$ - = 2 (a = 1. b = -8. c = -12). In this particular example, the equation has to be expressed in the general quadratic form before the values of a, b and c be identified.

1.2.1. Techniques for solving quadratic equations

Solving a quadratic equation involves finding the values of the variable. There are at most two values (solutions or roots). The solutions can be obtained in a number of ways which-include the following:

i. Algebraically, in which case two techniques can be used namely;
  - using the formula; and
  - by factorisation.

ii. Graphically, which involves construction of a graph and the roots identified from the graph.

(a) Solving the quadratic equations by factorization

This method can only be applied to quadratic equations which have two factors and therefore, have to be factorised (expressed in factor form) before solving them. A quadratic equation is a product of two linear factors.

If(x + p)(x +q) = 0

Then, either (x + p) = 0 or (x + q) = 0

Thus, x = -p orx = -q.

The quadratic equation $ax^2 + bx + c = 0$ has factors if there exist two numbers p and q, such that pq = ac and p + q = b then $ax^2 + bx + c \equiv ax^2 + (p + q)x + c \equiv ax^2 + px + qx + c$.

The adjacent terms will have common factors and therefore factorisation will proceed after pairing the adjacent terms as illustrated below:

$ax^2 + bx + c \equiv (ax^2 + px) (+ qx + c)$

Example 1. 10

Given the quadratic equations:

i) $x^2+7x+12 = 0$

ii) $3x^2 + 2x - 8 = 0$

Required

Solve the quadratic equations by factorization method.

Solution

Given x2+7x+12=0

Note that a =1, b = 7, c = 12 and a x c = 12

Since 3 x 4 = 12 and 3 + 4 = 7

Then x2 +7x +12 = x2 + (3 + 4)x + 12 =0

$x^2 +3x + 4x + 12 = 0$

$x(x+3)+4(x+3) = 0$

$(x + 3)(x + 4) = 0$

Either $(x + 3) = 0$ or $(x + 4) = 0$

.-. $x = -3$, or $x = -4$

i. Given $3x^2 + 2x - 8 = 0$,

Note that a =3, b = 2, c =-8 and ac = -24

Since -4 x 6 = -24 and -4 + 6 = 2

From $3x^2 + 2x - 8 = 0$

Then $(3x^2 + -4x) (6x - 8) = 0$

$x(3x-4) + 2(3x-4) = 0$

$(3x-4)(x + 2) = 0$,

Either: $3x - 4 =0$ or $x + 2 = 0$,

$x = \frac{44}{33}$ or $x = -2$.

(b) Solving by use of the general formula

For any quadratic equation $ax^2 + bx + c = 0$, its solutions are obtained from the general formula=x $= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Note:

When $b^2= 4ac$ there are two distinct solutions.
When $b^2= 4ac$ there is only one repeated solution.
When $b^2<=4ac$ there is no real solution.

Example 1.11

Given the equation $2x^2 -11x + 12 =0$.

Required

Solve by using the general quadratic formula.

Solution

$2x^2- 11x + 12=0$

From $x \frac{-b \pm \sqrt{b^2-4ac}}{2a} \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Where a = 2, b = -11,c = 12;

Then $x = \frac{-(-11) \pm \sqrt{(-1l)Z-4xZ*12}}{2x2} \frac{-(-11) \pm \sqrt{(-1l)Z-4xZ*12}}{2x2}$

$X=\frac{11 \pm \sqrt{25}}{4} \frac{11 \pm \sqrt{25}}{4}$

$x = \frac{6}{4} = \frac{36}{24} = \frac{3}{2}$ or $\frac{1616}{4\ 4} = 4$

The solutions of the equation $2x^2 -11x + 22 =10$ are x =1.5 and x = 4.

Example 1. 12

The area of a business premise x km^2 is given by the equation:

$$x + \frac{5}{2} = \frac{7}{2x}$$

Required

Find the area of the business premise.

Solution

Given that $x + \frac{5}{2} = \frac{7}{2x} x + \frac{5}{2} = \frac{7}{2x}$

Multiplying all through by 2x:

$2x^2 + 5x = 7$

Then $2x^2 + 5x - 7 = 0$; a = 2 b= 5 and c = -7.

Using the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{}$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4x2x - 1}}{}$$

$$x = \frac{- s \pm \sqrt{s^2}}{4} = \frac{-s + 9}{4} \; or \; \frac{-5 - 9}{4}$$

$$\therefore x = 1 \; or - 3.5$$

Hence, the area of the business premise is $1km^2$.

(c) Solving quadratic equations graphically

Solving the equation $ax^2 + bx + c = 0$ graphically involves:

i) Plotting a smooth curve $y = ax^2 + bx + c$, in the given range of x values.

ii) The solutions (or the roots) are obtained at the points where the curve crosses the x-axis, that **is,** the values of x when y = 0.

Example 1.13

Given the quadratic equations:

$2x^2 + 3x-6 = 0$

$5 + 8x - 3x^2 = 0$

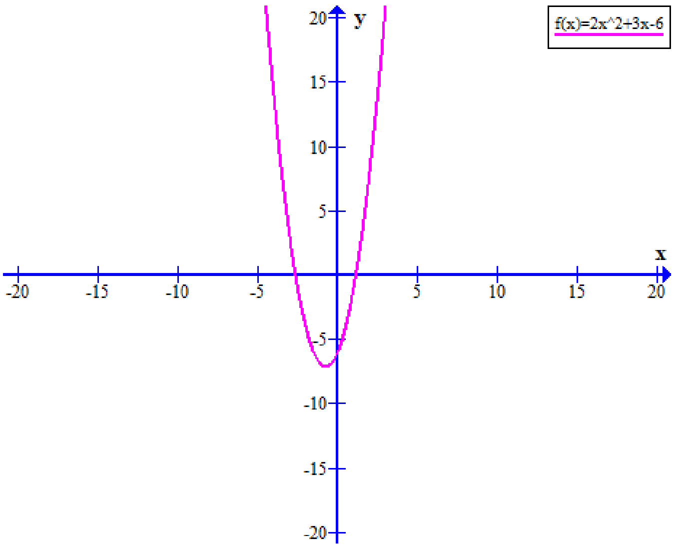
Required:

Solve the quadratic equations graphically for $-3 \leq x \leq 2$.

Solution

i) Considering $y = 2x^2 + 3x -6$

x	-3	-2	-1	0	1	2
$2x^2$	18	8	2	0	2	8
3x	-9	-6	-3	0	3	6
-6	-6	-6	-6	-6	-6	-6
y	3	-4	-7	-6	-1	8

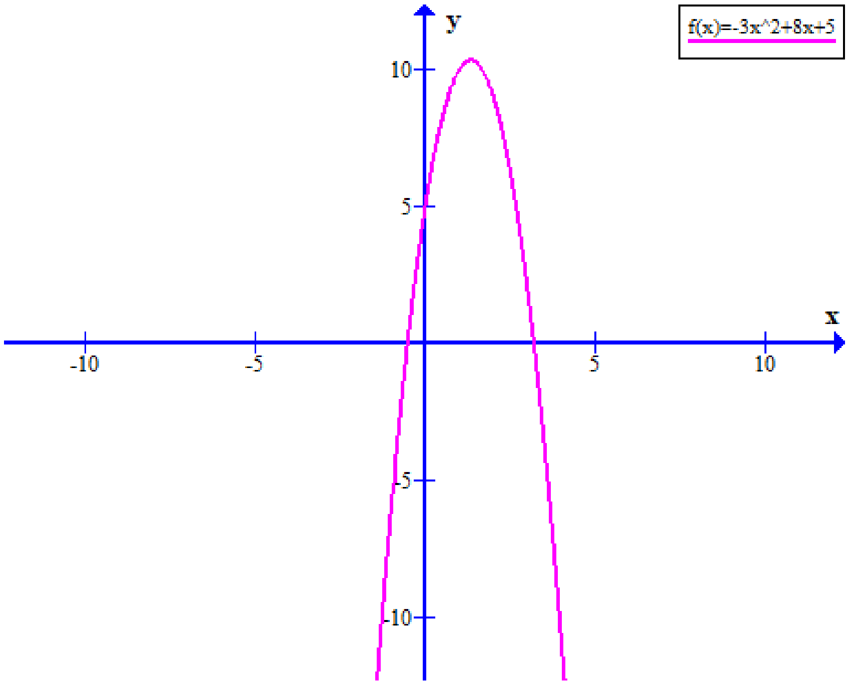


From the graph the curve crosses, the x-axis at -2.6 and 1.2.

Thus, the roots of the equation are x = -2.6 and x = 1.2.

i) Considering $y = 5 + 8x - 3x^2$

X	-1	0	1	2	3	4
5	5	5	5	5	5	5
8x	-8	0	8	16	24	32
$-3x^2$	-3	0	-3	-12	-27	-48
Y	-6	5	10	9	2	-11



From the graph the curve crosses the x-axis at -2.6 and 1.2.

.-.The roots of the equation are x = -2.6 and x = 1.2

(d) Solving quadratic equations algebraically

Example 1.14

Solve the equation $5x^2 +9x =2$ by factorisation.

Solution:

$5x^2 +9x -2 =0$

$5x^2 +10x-x-2 =0$

5x(x +2)- (x+2)=0

(x+2)(5x- 1)=0. So, x=-2 and x= $\frac{11}{55}$

Example 1.15

Solve the x $\frac{1}{x-1}+\frac{2}{3}=\frac{2 \quad 1}{x-3x-1}+\frac{2}{3}=\frac{2}{x-3}$ using the general formula.

Solution:

Clearing the fractions by multiplying throughout by 3(x-1)(x-3)

3(x-3)+2(x-1)(x-3) = 6(x-1)

$3x -9 +2(x^2 -4x +3) =6x-6$

Collecting like terms $2x^2 -11x +3=0$

Using the formula $x=\frac{-b\pm\sqrt{(b^2 4ac)}}{2a}x=\frac{-b\pm\sqrt{(b^2 4ac)}}{2a}$and putting a=2, b=-11. c=3,

$$x = \frac{11 + \sqrt{121 - 24}}{4}$$

$$= \frac{11 \pm x\sqrt{97}}{4} = \frac{11 \pm x\sqrt{97}}{4} = \frac{11 \pm 9.849}{4} \frac{11 \pm 9.849}{4}$$

So, $x = \frac{11+9.849}{4} = 5.212 x = \frac{11+9.849}{4} = 5.212$

Or $x = \frac{11+9.849}{4} = 0.288 x = \frac{11+9.849}{4} = 0.288$

So, the solution is x= 5.212 or x= 0.288

(e) Method of completing the square

The method involves expressing the general quadratic equation $ax^2+bx+c=0$ in the form $a(x + p)^2 + q = 0$.

The necessary steps include:

1. Remove the constant to the right-hand side, as it does not help in finding the square.
2. Divide throughout by the coefficient of x^2 if this is not unity.
3. Add to each side the square of half the coefficient of x.
4. Find the square root of both sides.

Example 1.16

Solve the quadratic equation $x^2 - 2x - 3 = 0$ by completing the square.

$X^2 - 2x = 3$

$X^2 - 2x + (-1)^2 = 3+1$

$(x - 1)^2 = 4$

$X-1=\pm 2$

$X= 1\pm 2$.-. x=-1 and x=3

1.2.1.1. Self-test questions

Question 1.1

Given the following simultaneous equations:

i) x + y = 5
2x - 3y =-5

ii) 4e + t=-16
e -3t = -17

iii) 2w - 8 = u
3w + 54 = -2u

iv) 5q = 4p +2
12p +10q = 9

Required:

Solve the given simultaneous equations.

Solution*:*

i. x = 2, y = 3
ii. e = -5, t = 4
iii. w = -10. u = -12
iv. $p = \frac{2}{4}\frac{3}{5} p = \frac{2}{4}\frac{3}{5}$
v. P=**V 5**

Question 1.2

Three people had their balances in their bank accounts x, y and z, in millions of Rwandan Francs at the close of business on a certain day given by the simultaneous equations.

a) x + y + 2z = 2
b) 3x + y - 4z = -7
c) x + 4y-6z = 10

Required:

Find the balance of each person.

Solution:

i. Frw 3 million.
ii. Frw 4 million.
iii. Frw 0.5 million.

Question 1.3

A soft drink company used 12 Scania trucks and 30 Isuzu trucks to deliver 4,800 crates of soda in the morning to a certain depot. In the afternoon 6,300 crates were delivered to another depot using 15 Scania trucks and 40 Isuzu trucks.

Required:

A certain depot hired 25 Scania trucks and 25 Isuzu trucks. Find the number of crates that were delivered.

Solution:

The number of crates delivered was 5,500.

Question 1.4

A restaurant serves its clients as first class, second class and third class each attracting a tip of Frw 7,000, Frw 5,000 and Frw 4,000respectively. On a certain day the restaurant received 100 clients both at lunch and dinner time. The total amount collected in tips at lunchtime was Frw 630,000. Compared with lunch time, the clients served at dinner as first class were less by 20, those served as second class more by 10 and those served as third class were twice as many.

Required:

Calculate the number of clients served in each class at lunch time.

Solution:

The number of clients served was 70, 20 and 10 clients as first class, second class and third class, respectively.

Question 1.5

Given the following quadratic equations:

a) x2 + 7x + 10 = 0

b) y2 + 3y-10 = 0

c) n2-n-12 = 0

d) 2p2 + 3p-2 = 0

e) 5t2-13t+6 = 0

Required:

Solve the given quadratic equations using the:

method of factorisation: and general formula.

Solution:

a) x = (2, 5)

b) Y = (-2,5)

c) n = (-3.4)

d) p = (-2,0.5)

t = (0.6,2)

Question 1.6

Given that $4x^2 + 4x - 15 = 0$ and that $-4 \leq x \leq 4$ solve the quadratic equation given graphically.

Solution:

x = (-2.5, 1.5).

Question 1.7

Solve the equation $3x^2 - 5x + 1 = 0$ by completing the square.

Solution:

x = (1.15, 0.52).

Question 1.8

Solve x^2 - x - 1 =0 by completing the square.

Solution:

x = (- 0.618, 1.618).

Question 1.9

The cost of a square carpet is Frw 248,000 and the cost of another square carpet whose side is 3 m longer than that of the first is Frw 387,500. The cost per m^2 is the same for both carpets.

Required:

Find the area of each carpet.

Solution:

Area = 9 and 36.

CHAP 2.

FUNCTIONS AND GRAPH

2.1. Study objectives

At the end of this chapter, should be able to:

express variable using functional notation; and

Represent variable function on the graph.

2.2. Functions and graphs

A fundamental idea in mathematics and its application is that of a function; which tells how one quantity depends on others. In applications of mathematics, functions are often representatives of real phenomena or events. Functions therefore are models. Obtaining a function to act as a model is commonly the key to understanding business in many areas.

Functions may be represented by formulae. There are **a** number of common ways in which functions are presented and used. We shall consider functions given by formulae, since this provides a natural context for explaining how a function works.

2.2.1. Functions of one variable

If you get a job that pays Frw 700 per hour, the amount of money M (in Rwandan Francs)that you earn depends on the number of hours (h) that you work, and the relationship is given by a simple function:

Money = 700 x hours worked

M = 700h Rwandan Francs

The formula M=700h shows that the money M that you earn depends on the number of hours worked. We say that M is a function of h.

In this context, h is a variable whose value we may not know until the end of the week. Once the value of h is known, the formula M=700h can be used to calculate the value of M. To emphasise that M is a function of h, it is common to write M=M(h),so that M(h)=700h. For example, if you work 30 hours, then the function M(30) is the money you would earn. To calculate the amount earned, you need to replace the formula by 30. Thus, M = M(30) = 700 x 30 = 21,000 Rwandan Francs .

Note:

i. It is important to remember that h is measured in hours and M is measured given is Rwandan Francs. The function/formula is not useful unless you state in words the units you are using.

ii. ii) We would also use different letters or symbols for the variables. Whatever letter/symbol used, it is critical that you explain in word what they mean.

Example 2.1

A bicycle covers a distance in 20 seconds. The speed of the bicycle is given by s =d/20=0.05d ms^{-1}, where s = speed (in ms^{-1}). d = distance (in metres, m). If the distance covered by the bicycle is 20 m, then the speed = 0.05x30 = 1.5 $ms^{-1.}$ If dicreases, the speed goes up and when d increases, the speed goes down. The speed is a function of distance (when time is constant). We get only one value of for each value of d.

2.2. 2. Functional notation and substitution

We normally write functions as f(x) and read as "function f of x". We could use other letters such as g, p, H or h and write the function of x as g(x), p(x), H(x) or h(x).

Given f(x) = 3x +5, the value of this function f(x) when x=0 is written as f(0). The values are obtained by substituting.

Example 2. 2

a) Given f(x) = 3x + 5, find i) f(0), ii) f(2) and iii) f(-2).

b) Given that h(x) =2x2 +5x +3, find i) h(0). ii) h(-2) and iii) h(3).

c) Given that G(t) = 2 +3t - 5t2 +t3, find i) g(-1) and g(2).

d) If h(x) = 5 - 2x, find the value of x for which the function is zero.

Solution:

a) i) f(0) =3(0) + 5 = 0 + 5 = 5

ii) f(2) =3(2)+5 = 6 + 5=11

i) f(-2) = 3(-2) + 5 = -6 +5 = -1

b) i) $h(0) = 2(0)^2+ 5(0)+3 = 3$

ii) $h(-2) = 2(-2)^2 + 5(-2) +3 =1$

ii) $h(3) = 2(3)^2+ 5(3)+3 = 36$

c) i) $G(-1) = 2+3(-1)-5(-1)^2-2(-1)^3 = -4$

ii) G(2) = 2+3(2) - 5(2)2 - 2(2)3 = -28

d) h(x) = 5-2x = 0

-2x = - 5

x = 2.5

Example 2. 3

a) If f(x) = 5x -3. find f(x+2).

b)If y = g(x) = 3x - 7. find g(y) in terms of x.

Solution:

a) Replace x by the assigned value (x+2) in the given equation:

f(x+2) = 5(x+2) - 3 = 5x +10-3 = 5x+ 7

b) Replace x by the assigned value y = 3x - 7 in the equation:

g(y) = 3y-7

= 3(3x-7)-11 = 9x -21 - 11 = 9x – 32

2.2.3. Defining the graph of a function

The graph of a function is a set of all points whose co-ordinates (x,y) satisfy the function y =f(x). This means that for each x-value there is a corresponding y-value which is obtained when we substitute into the expression for f(x). So, the graph of a function is a special case of the graph of an equation.

2.3. Types of functions

2.3.1. Linear functions

Equations that can be written in the form y = ax + b where a and b are constants are called linear functions. For linear functions of the form y = ax + b, the power of x is always 1. If we plot such a function against x, we always get a straight line.

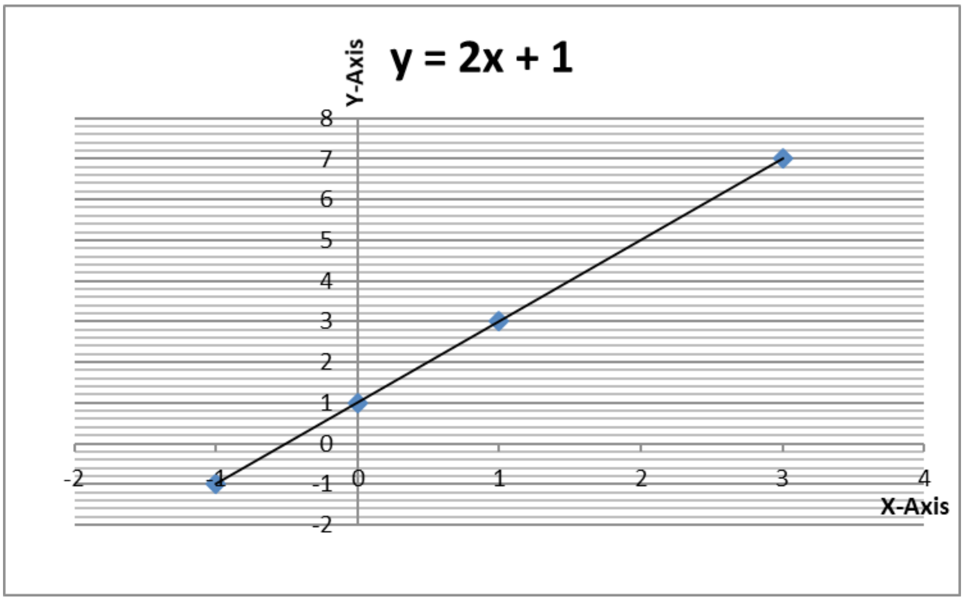
Where the number of possible number of points for the graph of a function is not given, the following procedures should be followed:

i. Select a few values of x (at least 5).
ii. Obtain the corresponding values of the function and enter them into a table
iii. Plot these points and join them with a smooth curve.

Example 2. 4

Draw the graph of the function f(x) = 2x + **1**

X	**-1**	**0**	**1**	**2**	**3**
Y	**-1**	**1**	**3**	**5**	**7**



2.3.2. Constant functions

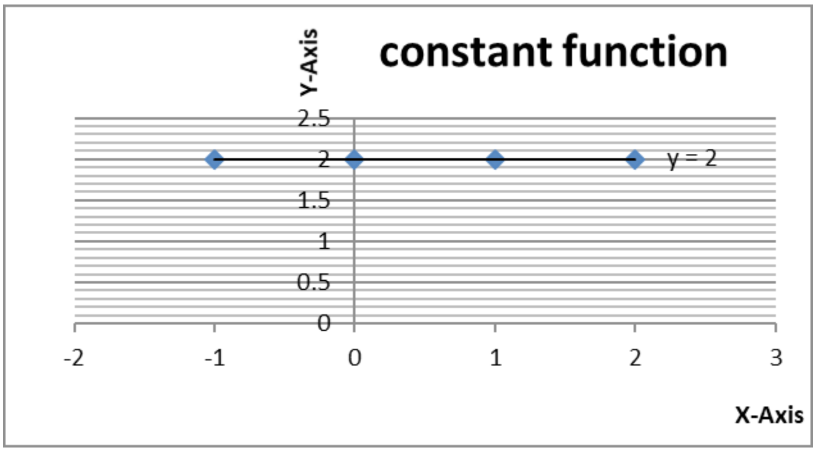
When a = 0, the function y = ax + b becomes y = f(x) = b which is special function called constant function.

Example 2. 5

Draw the graph of f(x) = 2

First find some possible values of x for which f(x) = 2

X	-1	0	1	2	3
Y	2	2	2	2	2



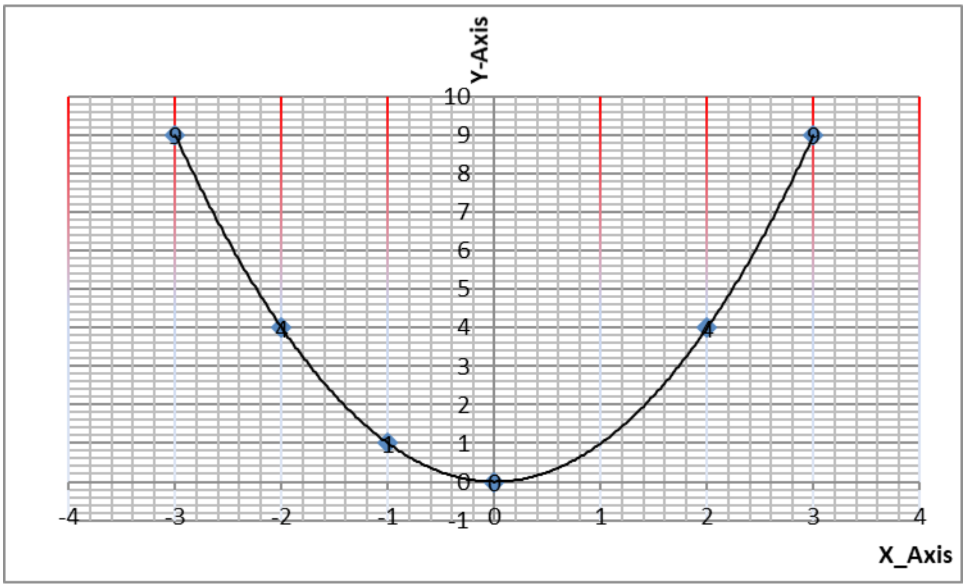
2.3.3. Quadratic functions

The expression f(x) = $ax^2 + bx + c$ where a, b and c are constants is called a quadratic function of x. The highest power of-the variable x is 2. If values of x are substituted, and the values of f(x) obtained are plotted against x, the result will be a curve.

Example 2.6

Draw the graph of the function $f(x) = X^2$ for x = {-3, -2, -1,0,1,2, 3}

X	-3	-2	-1	0	1	2	3
f(x)	9	4	1	0	1	4	9



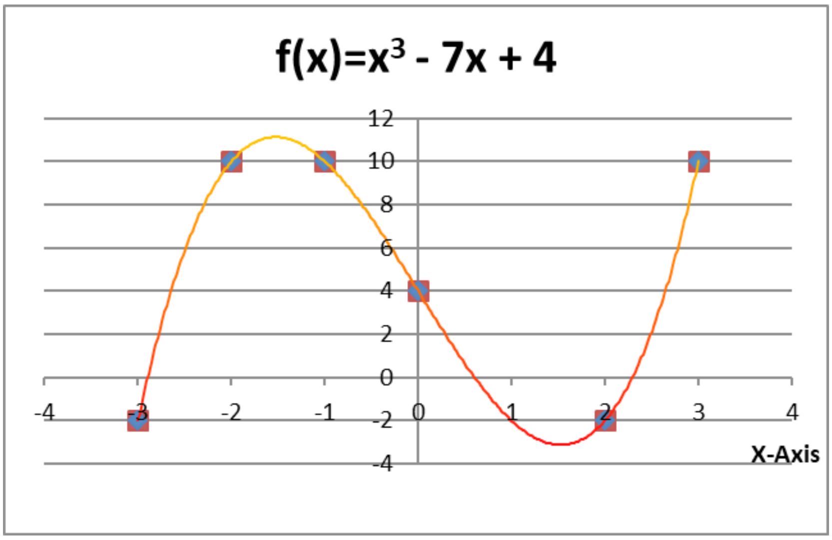
2.3.4. Polynomial functions

Functions of the form $a+bx+cx^2+dx^3+.....$, for constants a,b, c and d.

Example 2.7

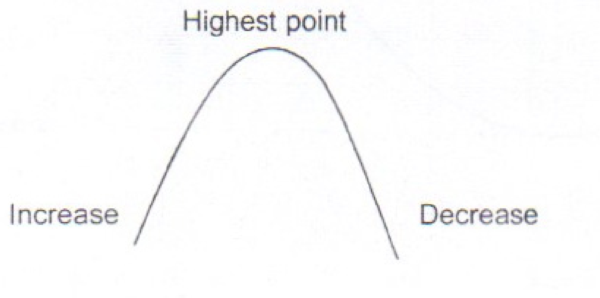
The function $f(x) = x^3 - 7x + 4$

X	-3	-2	-1	0	1	2	3
f(x)	-2	10	10	4	-2	-2	10



In sketching curves, the direction in which the curve moves is important. The curve may:

i. Increase, reaches a highest point and then reduces. When this happens, a highest point of a curve is observed:



ii. Decrease, reaches a lowest point and then increases When this happens, a lowest point is observed:

At either highest or lowest point, the gradient of the curve is always zero. In order to find the highest or lowest point of a given curve, the derivative of the curve is obtained and is equated to zero. For the given value of x obtained, substitute it in the original curve to get the point which is either minimum or maximum.

Example 2.8

Given the function $f(x) = 4x - 2x^2$, required to sketch the curve.

Solution

$\frac{dydy}{dxdx}$= 4 - 4x=0 at either minimum or maximum point.

∴∴ 4x =4, x=1

When x=1, f(x) = 4(1) -2(1 $)^2$ = 2. Thus, the point (1,2) is either maximum or minimum of the function. Find the points where the curve cuts the axes.

- At x-axis (y=0)

  4x - 2x2 =0

  2x(2 -x)=0

And x=0 and x= 2. Hence, the points at which the curve cuts the x-axis are

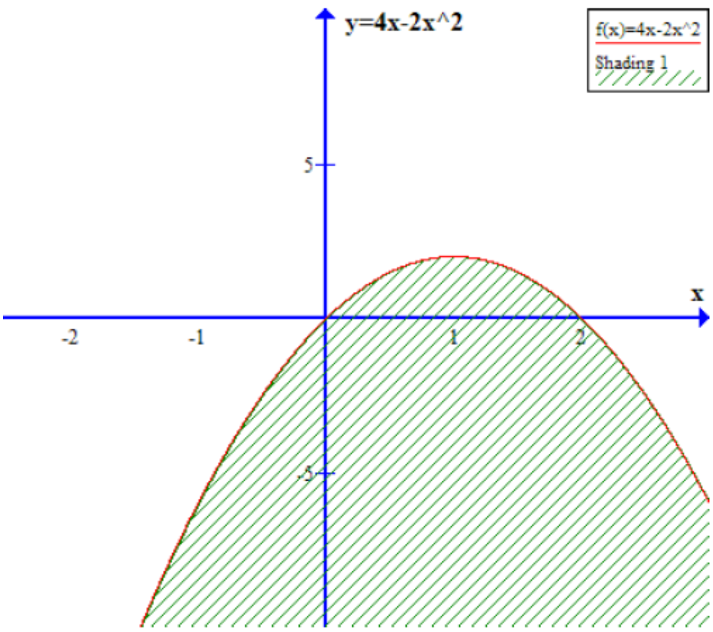
(0,0) and (2,0).

At y-axis (x=0)
4(0) -2(0)2 =0

The curve cuts y-axis at (0,0).

The points to be plotted and joined are (0,0), (1,2) and (2,0).

The sketch of the function $f(x)=4x -2x^2$



Thus, the point (1,2)Is a maximum-point.

Example 2. 9

Given the curve of the function $f(x) =x^2 +4x+3$

Required:

i) Sketch its curve.

ii) State the maximum or minimum point.

Solution:

The function f(x) $=x^2+4x+3$

$\frac{dydy}{dxdx}$= 2x+4=0 at either maximum or minimum point.

Thus, 2x=-4

And x=-2

When x=-2, f(-2)= (-2)2 +4(-2)+3 = -1

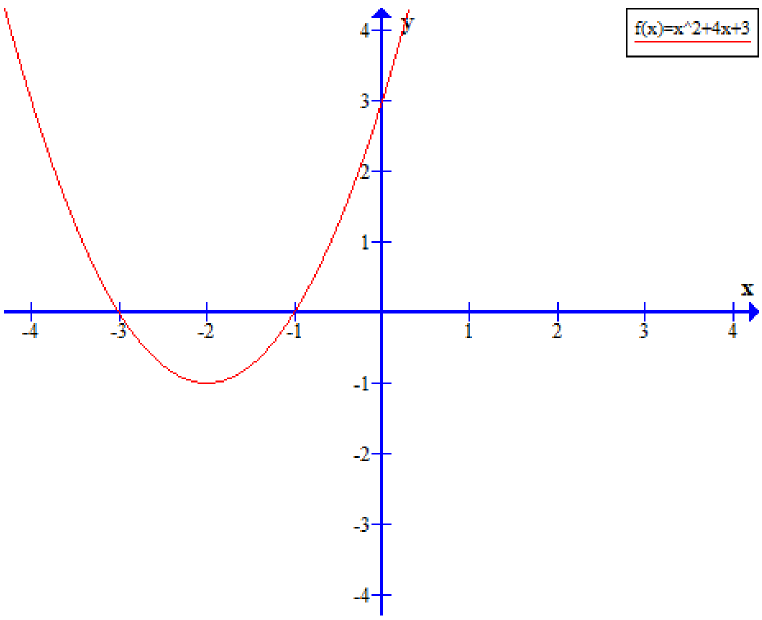
The point (-2,-1) is either maximum or minimum.

When x=0, f(0)=0+0+3=3. Hence the function cuts y-axis at (0,3).

When y=0, x2 +4x+3=0

(x+1)(x+3)=0, x=-1 and x=-3. Function cuts the x-axis at (-1,0) and (-3.0).

The points to plotted and joined are (-3,0), (-2.-1) and (0,3)



The point (-2.-1) is the minimum point.

Self-test questions

Question 2.1

If f(x) = 10x - 7, find f(2).

Solution: The answer is 13.

Question 2.2

If h(x) = 4x - 9, find h(0).

Solution: The answer is -9.

Question 2.3

Let g(y) = $\frac{2y+12y+1}{7y-57y-5}$. Find g(-6).

Solution: The answer is $\frac{1111}{4747}$.

Question 2.4

If p(x) = $\frac{3 \quad 3}{10-x \quad 10-x}$ find an expression in x for p(3x+5).

Solution: The answer is $\frac{3 \quad 3}{5-3x5-3x}$

Question 2.5

Given that z = h(x) =9 - 2x, find h(z).

Solution: The answer is 4x - 9.

Question 2.6

Draw the graphs of each of the allowing functions for x= $\{-3,-2,-1,0,1,2,3\}-3,-2,-1,0,1,2,3\}$

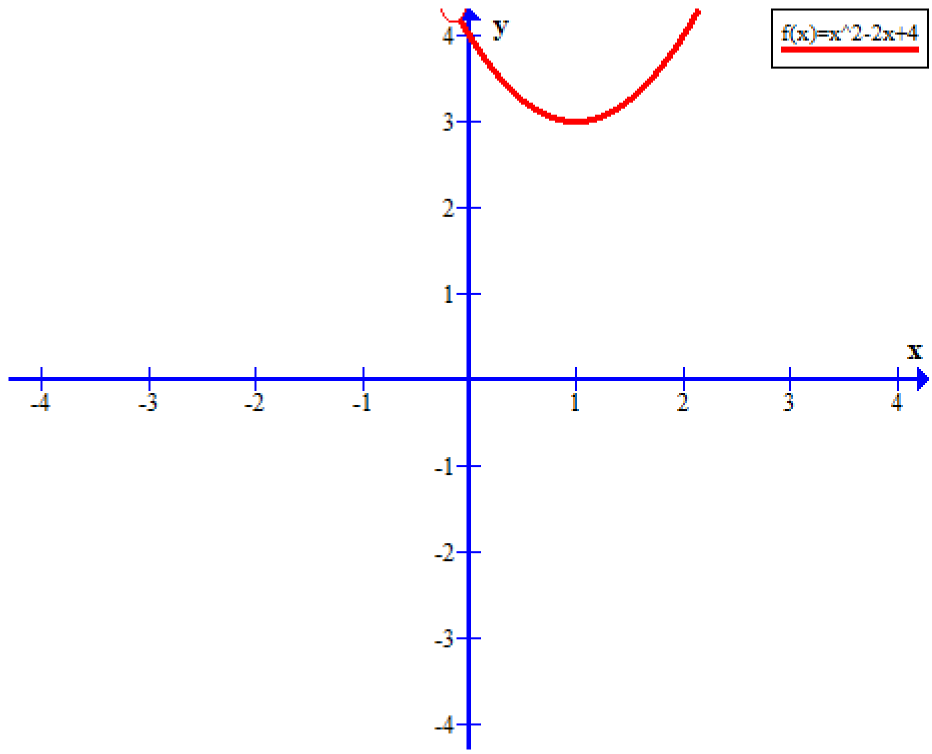
i) y = x2-2x + 4

ii) y = -2x2 + x + 4 ^

iii) y = x3-2x2+3x-5

Solution:

i)



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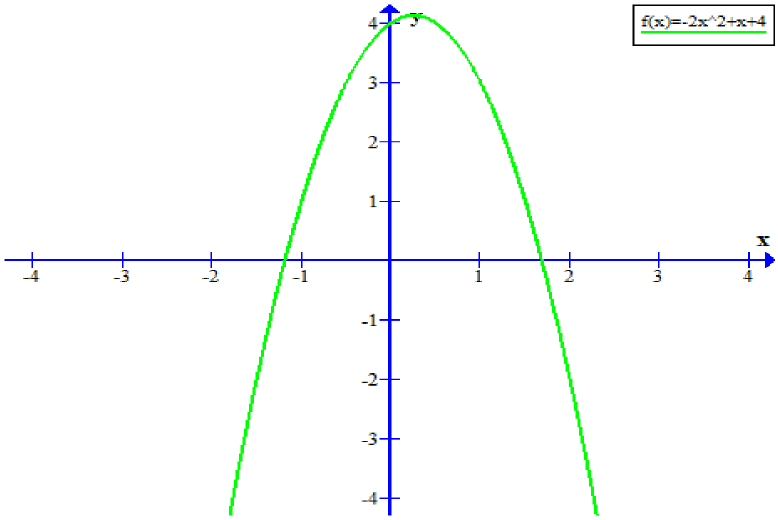
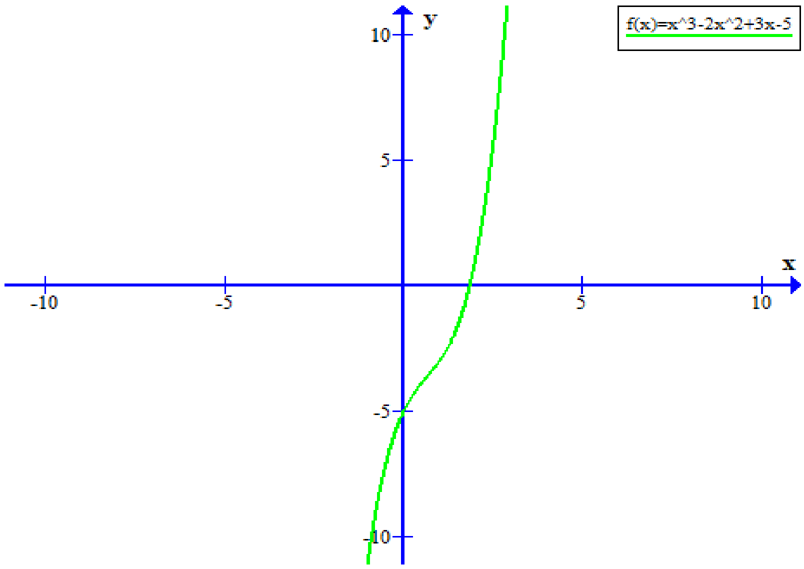


iii)



Question 2.7

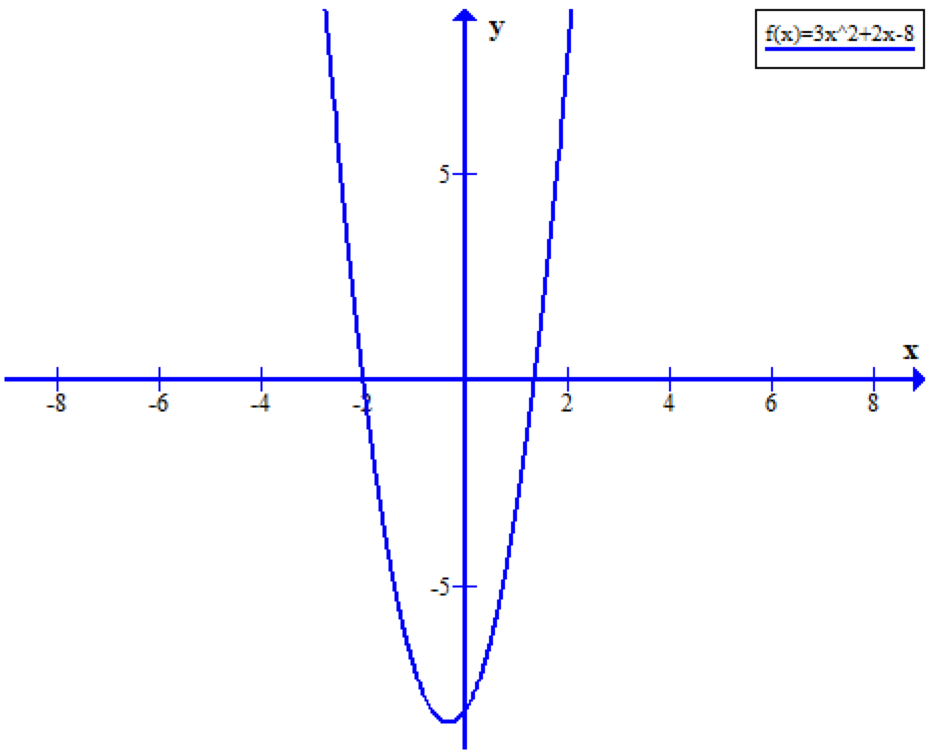
Given the values of x = {-3, -2, -1,0, 1,2, 3}, required to graph each of the functions:

i) $f(x) = 3x^2 + 2x - 8$

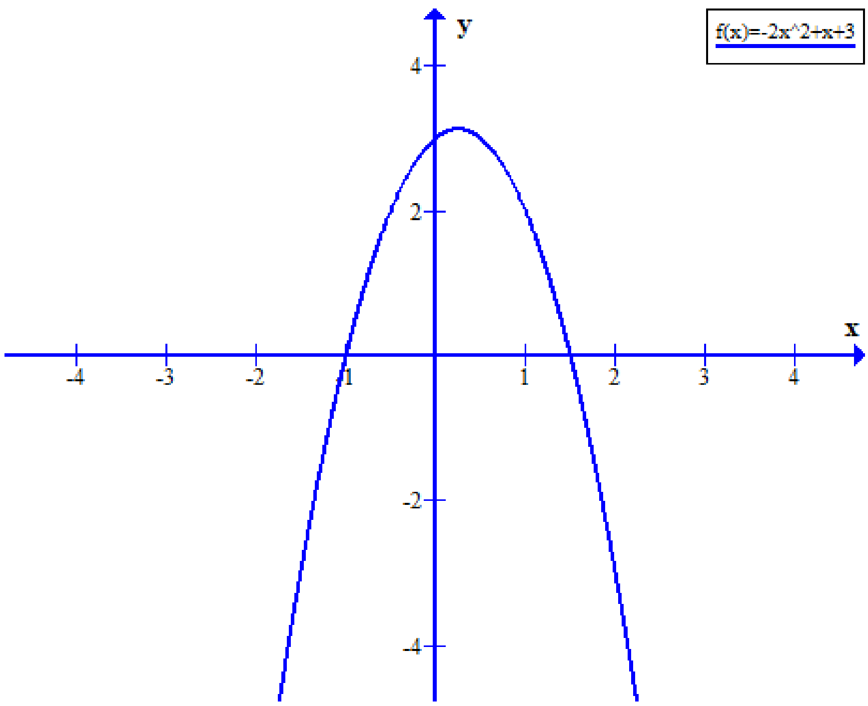
ii) $f(x) = -2x^2 + x + 3$

Solution:

i)



ii)



Question 2.8

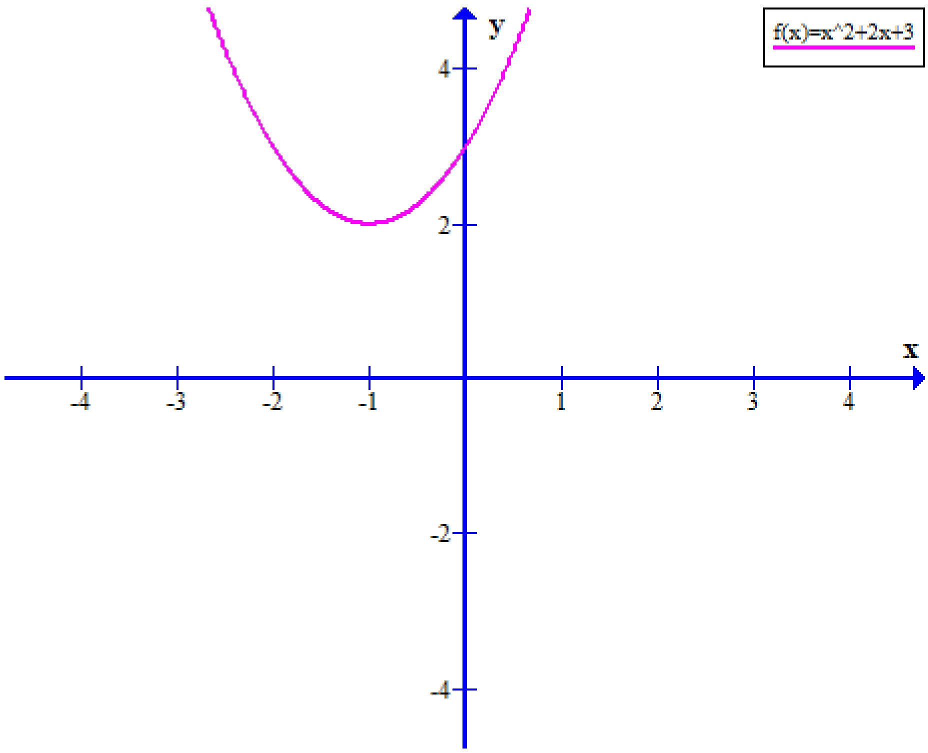
For the function $y = x^2 + 2x + 3$, required to:

i) Sketch the curve of the function; and

ii) find the minimum or maximum point.

Solution:

i)



Question 2.9

Given the function $y = 2x^2 - 5x - 2$, required to find the points at which the curve cuts the axes:

Point at which $\frac{dydy}{dxdx}$ = 0; and Sketch the curve.

Solution:

Curve cuts x - axis at (2,0) and(0.5 0) and y - axis at (0.2)

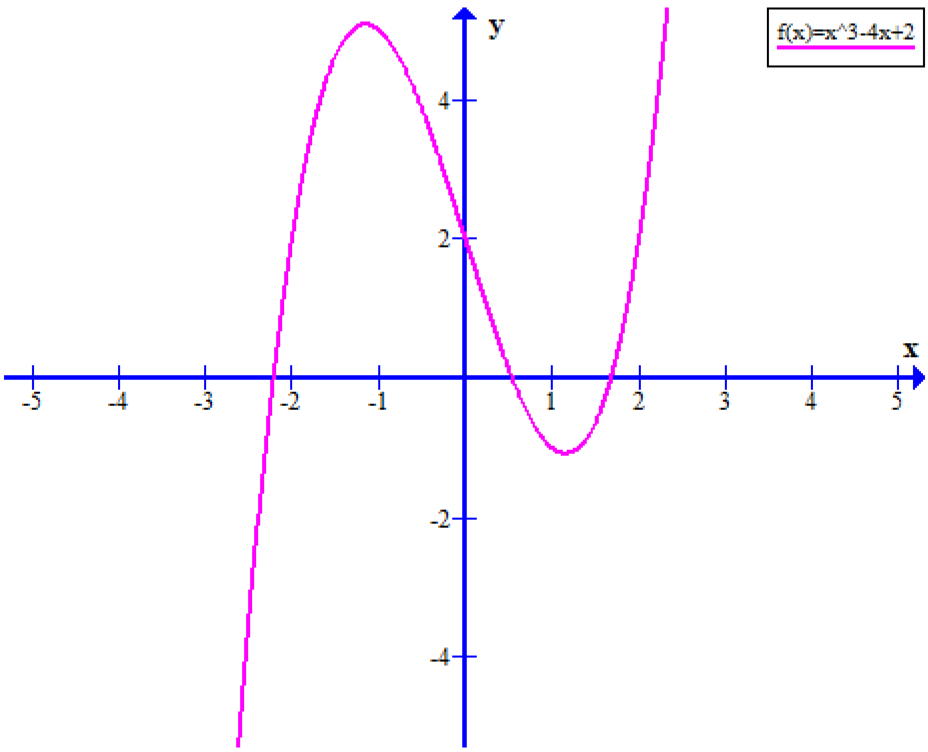
Point at which $\frac{dydy}{dxdx}$= 0 is ($\frac{5}{4}$, $-\frac{95}{44}$, $-\frac{9}{4}$)

Sketch of the curve to be supplied later.

Question 2.10

Graph the curve of the function $f(x)= x^3 - 4x +2$ for the values of x from -3 to +3.

Solution:



2.3.5. The Exponential Function

An exponential function has at a least one term for the independent variable as part of an exponent or power e.g. $y = 10^{2x}$

=> 10 is called the base at the function

-> 2x is the exponent or power

Important classes of exponential functions in business are those which have naturally occurring constant, e as their base i.e. $y = ae^{kx}$, where a, e and k are constants and specifically, e is a specific constant associated with continuous growth or continuous decay.

$$e = \text{Lim} \left(1 + \frac{1}{x}\right)^x \left(1 + \frac{1}{x}\right)^x \quad \text{as } x \to \infty$$

lim means Limit and $\to$ means approaches or tends to.

Approximations of e

x = 1 $\quad e \left(1 + \frac{1}{1}\right)^1 \left(1 + \frac{1}{1}\right)^1 = 2$

x = 2 $\quad e \left(1 + \frac{1}{2}\right)^2 \left(1 + \frac{1}{2}\right)^2 = 2.25$

x = 10 $\quad e \left(1 + \frac{1}{10}\right)^{10} \left(1 + \frac{1}{10}\right)^{10} = 2.5937$

x = 100 $\quad e \left(1 + \frac{1}{100}\right)^{100} \left(1 + \frac{1}{100}\right)^{100} = 2.7048$

x = 1000 $\quad e \left(1 + \frac{1}{1000}\right)^{1000} \left(1 + \frac{1}{1000}\right)^{1000} = 2.7169$

Illustration

Sketch the following 2 functions on the same graph

(1) y = $e^x e^x$ (2) y = $e^{-x} e^{-x}$

X	-3	-2	-1	0	1	2	3
e^x	0.05	0.14	0.37	1	2.72	7.39	20.09
e^{-x}	20.09	7.39	2.72	1	0.37	0.14	0.05

20-

18-

12-

10-

8-

6-

4-

2-

-3 -2 -1 1

Notes

1. For most application, exponential functions have either time or space as the independent variable e.g. population level = f (time)

Population level = f (area of distance covered)

2. Equal changes in the independent variable for an exponential function results in constant % change in the value of the dependent variables; this constant % change is the coefficient of the independent variable.

For y = ae^{kx}

=> a is the value of y when x = 0 a > 0

e.g Initial population, initial (purchase cost of an asset.............)

=> k is the constant % change per unit of x if k is positive then, it is a growth function but if k is negative, it is a decay function.

e.g. $y = 10e^{0.2x}y = 10e^{0.2x}$
Initial value of y is 10
Growth rate is 20% per unit of x e.g. per annum
$y = 25e^{-0.05x}$
Initial value of y is 25
Rate of decay/decrease is 5% per unit of x .... e.g. per m^3

2.3.6. Logarithmic Functions

A logarithmic is a power which a base must be raised in order to give a certain number i.e. a logarithmic is an exponent.

e.g. $2^3 = 8$

This is equivalent to $\log_2 8 = 3$ i.e 3 is the log to base 2 of the number 8

Equivalent Exponential and Logarithmic Forms

Exp. Form	Log Form
$10^2 = 100$	$\log_{10} 100\log_{10} 100 = 2$
$5^3 = 125$	$\log_5 125\log_5 125 = 3$
$4^3 = 64$	$\log_4 64\log_4 64 = 3$

Although logarithms can be taken to any base, the most commonly used bases are base 10 and base 2.

Further, base 10 logarithms are denoted "log" while base e logarithms are denoted "In" –> e^x (also known as natural logarithms e.g. log 100 = 2. And In 100 = 4.605 $\Longrightarrow e^{4.605} \Longrightarrow e^{4.605}$ = 100

Properties of Logarithms

1. Log uv = Log u + Log v
e.g Log 100 x 1000 = Log 100 + Log 1000
= 2 + 3 = 5
= Log 100000
2. Log $\frac{uu}{vv}$ = Log u – Log v
e.g. Log $\frac{10001000}{100\ \ 100}$
= log 1000 – log 100
= 3 – 2 = 1 = log 10
3. Log u^n = n log u
e.g. $\text{Log}10^2 = 2\log 10$
= 2 x 1 = 2
= log100
4. $\text{Log}_b b = 1$ Since $b^1 = b$

5. $\text{Log}_b 1 = 0$ Since $b^0 = 1$

6. $\text{Log}_b b^x = x$ $\text{Log}_b{}^b = x$
= since $\text{Log}_b b = 1$
∴ $\log_b b^x = x$

Applications of exponential and logarithmic functions

1. Growth processes
   - i) Population growth is exponential
   - ii) Spread of a contagious disease
   - iii) Growth in value of certain assets e.g. land
   - iv) Rate of inflation

2. Decay process
   - i) Asset depreciation e.g. computers and electronics generally.
   - ii) Decrease in purchasing power of the shilling.
   - iii) Decline in the rate of incidence of certain disease such as polio as medical research and technology advances.
   - iv) Decrease in the value of a share in the stock exchange as negative sentiments concerning it spread etc.

2.4. Application problems in cost, revenue and profit

Problem 2.1

Super Toys Ltd. (STL) manufactures and sells toys. "Super car" is one of their popular models. The marketing department has estimated the demand function for the model to be linear. If the price was fixed at Frw 570, the daily sales of the model would be 400 toys, whereas if the price was increased to Frw 820, the daily sales would drop to 200 toys.

Data from the production department indicate that the incremental cost of producing q toys of the model is given by the equation;

$\Delta C(q) = 2q - 570$ and that the daily fixed cost is Frw 1,100.

Required:

(i) The revenue functions if q toys are sold.

(ii) The total cost function.

(iii) The daily break-even number of toys

(iv) The point elasticity of demand when the demand is 110 toys. Interpret the economic meaning of your result.

Solution

(i) Demand slope = $\underline{570 - 820}$ = -1.25

400 – 200

Equation of demand

P – 570 = -1.25

q – 400

P = -1.25 (q – 400) + 570 = -1.25q + 1070

\Revenue, R = (1070 – 1.25q) q = 1070q – 1.25q²

(ii) Total cost, TC = ∫ (2q – 570) dq

= q² – 570q + C

C = fixed cost = 1,100

TC = q² – 570q + 1,100

(iii) Profit, π = 1070q – 1.25q² – q² + 570q – 1100

= -2.25q² + 1640q – 1100

At B.E.P, profit = 0 ⟹ -2.25q² + 1640q – 1100 = 0

$$q = \frac{-1640 \pm \sqrt{1640^2 - 4(-2.25)(-1100)}}{2(-2.25)}$$

q = 0.67 or q = 728

(iv) P = 10 70 -1.25q

$$\frac{dp}{dq} = -1.25 \; Þ \frac{dq}{dp} = \frac{1}{-1.25} = -0.8$$

When q = 110, p = 932.5

Point of elasticity, $E = \frac{p}{q} \times \frac{dp}{dp}$

= 932.5 x -0.8

110

= -6.78

Problem 2.2
Puda Development Company (PDC) is a small real estate developer operating in the Eastlands Valley. It has seven permanent employees whose monthly salaries are given below:

Employee	**Monthly salary (Frw)**
Managing Director	100,000
Manager, Development	60,000
Manager, Marketing	45,000
Project Manager	55,000
Finance Manager	40,000
Office Manager	30,000
Receptionist	20,000

PDC leases a building for Frw 20,000 per month. The cost of suppliers, utilities and leased equipment runs for another Frw 30,000 per month. PDC builds only one style house in the valley. Land for each house costs. Frw 550,000 and lumber, supplies and others run for another Frw 280,000 per house. Total labour costs amount to Frw 200,000 per house. The one sales representative of PDC is paid a commission of Frw 20,000 on the sale of each house. The selling price of the house is Frw 1,150,000.

Required:

i) Identify all the costs and deduce the marginal revenue and marginal cost for each house.

ii) Determine the monthly cost function; C(x), revenue function; R(x) and the profit function; P(x)
iii) Determine the break-even point for monthly sales of the houses.
iv) Determine the monthly profit if 12 houses per month are build and sold.

Solution
(i) **Salaries (Frw '000):**

100 + 60 + 45 + 55 + 40 + 30 + 20

= 350

Office lease and supply costs= 20 + 30= 50

Fixed cost= 350,000 + 50,000= 400,000

- Land, Material, labour and sales commission per house is the variable or marginal cost for the house. It is given as:

  = 550,000 + 280,000 + 200,000 + 20,000

  = <u>1,050,000</u>

- The selling price of Frw 1,150,000 is the marginal revenue per house.

(ii) Total cost function;

TC = VC + FC = 1,050,000x + 400,000

= 1,050,000 + 400,000 = 1,450,000

TR = 1,150,000 (x)

= 1,150,000x

Profit = TR – TC

= 1,150,000x – 1,050,000x – 400,000

= 100,000x – 400,000

(iii) Break even in number of houses;

At BEP TR = TC … substituting

$\Rightarrow$ 1,150,000x = 1,050,000x + 400,000

$\Rightarrow$ 100,000x - 400,000

$\Rightarrow$ x = 4 houses

(iv) The profit if 12 houses are built and sold is computed as equal to

= 100,000 x (12) – 400,000

= 1,200,000 – 400,000

= Frw 800,000.

Problem 2.3

The following information relates to M. Mutuma, a dealer in standard wooden tables:

M. Mutuma realized profits of Frw 12,000 from 7 tables, Frw 12,400 from 9 tables and Frw 11,300 from 4 tables sold respectively. M. Mutuma has approached you for assistance in forecasting future profits. The profit function is believed to be quadratic in nature.

Required:

(i) Derive the profit function.

(ii) The profit maximizing output and the maximum profit.

Solution

a (i) Profit function = $P=ax^2+bx+c$

$1200=a(7)^2 + b(7) + c \rightarrow 12000 = 49a+7b+c$

$12400 = a(9)^2 + b(9) + c \rightarrow 12400 = 81a + 9b+ c$

$11300 = a(4)^2 + b(4) + c \rightarrow \quad 11300 = 16a + 4b + c$

Solve the 3 equations simultaneously

$a = -\frac{20}{3}$ $\quad b = \frac{920}{3}$ $\quad c = 10,180$

$\therefore P = 10180 + \frac{920x}{3} - \frac{20x^2}{3}$ $\therefore P = 10180 + \frac{920x}{3} - \frac{20x^2}{3}$ is the required profit function.

ii) Profit maximizing output at maximum profit $\frac{dpdp}{dxdx} = 0$

FOC $\Longrightarrow \frac{920}{3} - \frac{40x}{3} = 0 \Longrightarrow x = 23 \Longrightarrow \frac{920}{3} - \frac{40x}{3} = 0 \Longrightarrow x = 23$

$\frac{40x}{3} = \frac{92040x}{2\ \ 3} = \frac{920}{2}$

x = 23

Maximum profit

$P = 10180 + \frac{920x}{3} - \frac{20x^2}{3}$

$= 10180 + \frac{920}{3}(23) - \frac{20(23)^2}{3}$

= <u>Frw13,706.67</u>

Problem 2.4

The data below relate to products A and B, manufactured by Sina Limited.

$q_1 = 2(P_2 - P_1) + 4q_1 = 2(P_2 - P_1) + 4$ is the demand function for product A

$q_2 = \frac{1}{4}P_1 - \frac{5}{2}P_2 + 52q_2 = \frac{1}{4}P_1 - \frac{5}{2}P_2 + 52$ is the demand function for product B

q_1 is the quantity of product A

q_2 is the quantity of product B

P_1 is the selling price of product A

p_2 is the selling price per unit of product B

The variable costs per unit are Frw 9 and Frw 12 for products A and B respectively.

Required:

i) The total revenue function of Manzo Limited.

ii) The total cost function of Mauzo Limited

iii) The total profit function of Mauzo Limited.

iv) The profit maximizing prices and quantities of products A and B

Solution

(i) $R_1 = P_1 q_1 = P_1(2P_2 - 2P_1 + 4)$

$R_1 = 2P_1P_2 - 2P_1^2 + 4P_1$

$R_2 = P_2 q_2 = P_2 \left(\frac{P_1}{4} - \frac{5}{2}P_2 + 52\right)$

$$R_2 = \frac{1}{4}P_1P_2 - \frac{5}{2}P_2^2 + 52P_2$$

Total revenue function, $R = R_1 + R_2$

$$R = \frac{9}{4}P_1P_2 - 2P_1^2 + 4P_1 - \frac{5}{2}P_2^2 + 52P_2$$

(ii) $C_1 = 9q_1 = 9(2P_2 - 2P_1 + 4)$

$C_1 = 18P_2 - 18P_1 + 36$

$C_2 = 12q_2 = 12\left(\frac{P_1}{4} - \frac{5}{2}P_2 + 52\right) = 3P_1 - 30P_2 + 624$

Total Cost Function, $C = C_1 + C_2$

$C = 660 - 12P_2 - 15P_1$

(iii) Total profit function, $\Pi = R - C$

$$\pi = \frac{9}{4}P_1P_2 - 2P_1^2 + 4P_1 - \frac{5}{2}P_2^2 + 52P_2 - 660 + 12P_2 + 15P_1$$

$$\pi = \frac{9}{4}P_1P_2 - 2P_1^2 + 19P_1 - \frac{5}{2}P_2^2 + 64P_2 - 660$$

(iv) At maximum profit $\frac{d\pi\, d\pi}{dP_1 dP_1}$ = 0 and $\frac{d\pi\, d\pi}{dP_2 dP_2}$ = 0

F.O.C: $\frac{d\pi\, d\pi}{dP_1 dP_1}$ = $^9/_4 P_2 - 4P_1 + 19 = 0$ .... (i)

$$\frac{d\pi}{dP_2} = \frac{9}{4}P_1 - 5P_2 + 64 \frac{d\pi}{dP_2} = \frac{9}{4}P_1 - 5P_2 + 64 = 0$$

$4P_1 - \frac{80}{9}P^2 + \frac{1024}{9} = 0$

$-4P_1 + \frac{9}{4}P_2 + 19 = 0$

$-\frac{239}{36}P_2 + \frac{1195}{9} = 0$

P_2=Frw20

P_1 =Frw16

S.O.C: $\frac{d^2\pi d^2\pi}{dP_1^2 dP_1^2}$ = - 4 (-ve) hence maximum

$\frac{d^2\pi d^2\pi}{dP_2^2 dP_2^2}$ = - 5 (-ve) hence maximum

∴Profit is maximized when P_1 =Frw16, P_2 = Frw20, q_1 = 2(20-16) + 4 = 12 units and q_2 = 16-5x20+52=6 units.

CHAP 3.

MATRICES

A matrix is a rectangular array of items or numbers. These items or numbers are arranged in rows and columns to represent some information. The position of an element in one matrix is very important as will be seen later; therefore an element is located by the number of the row and column which it occupies. The size of a matrix is defined by the number of its rows (m) and column (n).

For example A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and B = $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

are (2 x 2) and (3 x 3) matrices since A has 2 rows and 2 columns and B has 3 rows and 3 columns.
A matrix A with three rows and four columns is given by one of

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

or

$$A = \left(a_{ij}\right) \quad i = 1, 2, 3$$

$$j = 1, 2, 3, 4$$ *where i represents the row number whereas j represents the column number*

3.1. Types of matrices

3.1.1. Equal Matrices

Two matrices A and B are said to be equal, that is

$$A = B \qquad \text{or} \quad \left(a_{ij}\right) = \left(b_{ij}\right)$$

If and only if they are identical if they both have the same number of rows and columns and the elements in the corresponding locations in the two matrices should be the same, that is, $a_{ij} = b_{ij}$ for all i. And j.

Example 3.1

The following matrices are equal $\begin{pmatrix} 3 & 4 & 0 \\ 2 & 2 & 3 \\ 5 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 0 \\ 2 & 2 & 3 \\ 5 & 1 & 1 \end{pmatrix}$

3.1.2. Column Matrix or column vector

A column matrix, also referred to as column vector is a matrix consisting of a single column.

For example $x = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix}$

3.1.3. Row matrix or row vector

It is a matrix with a single row

For example $y = (y_1, y_2, y_3 \ldots\ldots\ldots y_n)$

3.1.4. Transpose of a Matrix

The transpose of an mxn matrix A is the nxm matrix A^T obtained by interchanging the rows and columns of A.

$A = a_{ij}$

The transpose of A i.e. A^T is given by

$$A^T = [a_{ij}]^T = [a_{ji}] A^T = [a_{ij}]^T = [a_{ji}]$$

xn nxm

Example 3.2

Find the transposes of the following matrices

$$A = \begin{pmatrix} 1 & 5 & 7 \\ 2 & 1 & 4 \\ 0 & 9 & 3 \end{pmatrix}$$

$$B = (b_1, b_2, b_3, b_4)$$

$$C = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Solution

$$\text{i. } A^T = \begin{pmatrix} 1 & 5 & 7 \\ 2 & 1 & 4 \\ 0 & 9 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 0 \\ 5 & 1 & 9 \\ 7 & 4 & 3 \end{pmatrix}$$

$$\text{ii. } B^T = (b_1, b_2, b_3, b_4)^T = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$\text{iii. } C^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T C^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T = (x_1, x_2, x_3)(x_1, x_2, x_3)$$

3.1.5. Square Matrix

A matrix A is said to be square when it has the same number of rows as columns e.g.

$$A = \begin{matrix} 2 & 5 \\ 3 & 7 \end{matrix}$$ is a square matrix of order 2

B = n × n is a square matrix of the order n

3.1.6. Diagonal matrices

It is a square matrix with zeros everywhere in the matrix except on the principal diagonal

e.g.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3.1.7. An identity or unity matrix

It is a diagonal matrix in which each of the diagonal elements is a positive one (1)

e.g.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2×2 unit matrix $\qquad$ 3×3 unit matrix

3.1.8. A null or zero matrix

A null or zero matrix is a matrix whose elements are all equal to zero.

3.1.9. Sub matrix

The sub matrix of the matrix A is another matrix obtained from A by deleting selected row(s) and/or column(s) of the matrix A.

$$\text{e.g, if } A = \begin{pmatrix} 7 & 9 & 8 \\ 2 & 3 & 6 \\ 1 & 5 & 0 \end{pmatrix}$$

$$\text{then } A_1 = \begin{pmatrix} 2 & 3 & 6 \\ 1 & 5 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 7 & 9 \\ 1 & 5 \end{pmatrix}$$

are both sub matrices of A

3.2. Operation on matrices

3.2.1. Matrix addition and subtraction

We can add any number of matrices (or subtract one matrix from another) if they have the same sizes. Addition is carried out by adding together corresponding elements in the matrices. Similarly subtraction is carried out by subtracting the corresponding elements of two matrices as shown in the following example

Example: Given A and B, calculate A + B and A – B

$$A = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & -13 & -6 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 12 & 4 & -7 & 3 \\ 0 & -4 & 10 & -4 \\ 7 & -3 & 7 & 9 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & -13 & -6 & 0 \end{pmatrix} + \begin{pmatrix} 12 & 4 & -7 & 3 \\ 0 & -4 & 10 & -4 \\ 7 & -3 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 18 & 3 & 3 & 8 \\ 3 & 0 & 12 & -9 \\ -2 & -16 & 1 & 9 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & -13 & -6 & 0 \end{pmatrix} - \begin{pmatrix} 12 & 4 & -7 & 3 \\ 0 & -4 & 10 & -4 \\ 7 & -3 & 7 & 9 \end{pmatrix} = \begin{pmatrix} -6 & -5 & 17 & 2 \\ 3 & 8 & -8 & -1 \\ -16 & -10 & -13 & -9 \end{pmatrix}$$

If it is assumed that A, B, C are of the same order, the following properties are fulfilled:

a) Commutative law: $A + B = B + A$

b) Associative law: $(A + B) + C = A + (B + C) = A + B + C$

3.2.2. Multiplying a matrix by a number

In this case each element of the matrix is multiplied by that number

Example 3.4

$$\text{If } A = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & 13 & -6 & 0 \end{pmatrix}$$

$$\text{then } (10)A = \begin{pmatrix} 60 & -10 & 100 & 50 \\ 30 & 40 & 20 & -50 \\ -90 & 130 & -60 & 0 \end{pmatrix}$$

3.2.3. Matrix Multiplication

3.2.3.1. Multiplication of two vectors

Let row vector A represent the selling price in Rwandan Francs of one unit of commodity P, Q and R respectively and let column vector B represent the number of units of commodities P, Q, R sold respectively. Then the vector product A ´ B will be equal to the total sales value

i. e. A ´ B = Total sales value

$$\text{Let } A = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix}$$

$$\text{then } \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix} = 400 + 1{,}000 + 1{,}800 = \text{Shs } 3{,}200$$

Rules of multiplication

i) The row vector must have the same number of elements as the column vector

ii) The first vector is a row vector and the second is a column vector

iii) The corresponding elements in each vector are multiplied together and the results obtained are added. This addition is always a single number

Going back to the example given before

$$A \times B = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix} = 4 \times 100 + 5 \times 200 + 6 \times 300 = \text{Shs}3{,}200, \text{ a single number}$$

3.2.3.2. Multiplication of two matrices

Rules

i) Multiplication is only possible if the first matrix has the same number of columns as the rows of the second matrix. That is if A is the order a×b, then B has to be of the order b×c. If the A×B = D, then D must be of the order a×c.

ii) The general method of multiplication is that the elements in row m of the first matrix are multiplied by the corresponding elements column n of the second matrix and the products obtained are then added giving a single number.

We can express this rule as follows

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $b = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$

Then $A \times B = D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{pmatrix}$

A = 2 x 2 matrix B = 2 x 3 matrix D = 2 x 3 matrix

Where

$d_{11} = a_{11} \times b_{11} + a_{12} \times b_{21}$

$d_{12} = a_{11} \times b_{12} + a_{12} \times b_{22}$

Example 3.5

$$\begin{pmatrix} 6 & 1 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 0 & 2 \\ 4 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 6\times3+1\times4 & 6\times0+1\times5 & 6\times2+1\times8 \\ 2\times3+3\times4 & 2\times0+3\times5 & 2\times2+3\times8 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 5 & 20 \\ 18 & 15 & 28 \end{pmatrix}$$

Example 3.6

Matrix X gives the details of component parts used in the make up of two products P_1 and P_2 matrix Y gives details of products made on each day of the week as follows:

Matrix X

Parts

$$\text{Products}\ \begin{matrix} \\ P_1 \\ P_2 \end{matrix}\begin{matrix} A & B & C \\ \left[\begin{matrix} 3 & 4 & 2 \\ 2 & 5 & 3 \end{matrix}\right. & & \end{matrix}$$

$$X = \begin{array}{c|ccc} & A & B & C \\ \hline P_1 & 3 & 4 & 2 \\ P_2 & 2 & 5 & 3 \end{array}$$

Matrix Y

Products

$$Y = \begin{array}{c|cc} & P_1 & P_2 \\ \hline \text{Mon} & 1 & 2 \\ \text{Tues} & 2 & 3 \\ \text{Wed} & 3 & 2 \\ \text{Thur} & 2 & 2 \\ \text{Fri} & 1 & 1 \end{array}$$

Use matrix multiplication to find the number of component parts used on each day of the week.

Solution

After careful consideration, it will be easy to decide that the correct order of multiplication is Y×X (Note the order of multiplication). This multiplication is compatible and also it gives the desired answer.

$$Y \times X = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 4 & 2 \\ 2 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1\times3+2\times2 & 1\times4+2\times5 & 1\times2+2\times5 \\ 2\times3+3\times2 & 2\times4+3\times5 & 2\times2+3\times3 \\ 3\times3+2\times2 & 3\times4+2\times5 & 3\times2+2\times3 \\ 2\times3+2\times2 & 2\times4+2\times5 & 2\times2+2\times3 \\ 1\times3+1\times2 & 1\times4+1\times5 & 1\times2+1\times3 \end{pmatrix}$$

5 x 2 matrix 2 x 3 matrix = 5 x 3 matrix

$$\begin{array}{c} \\ \text{Mon} \\ \text{Tues} \\ \text{Wed} \\ \text{Thur} \\ \text{Fri} \end{array} \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{pmatrix} 7 & 14 & 8 \\ 12 & 23 & 13 \\ 13 & 22 & 12 \\ 10 & 18 & 10 \\ 5 & 9 & 5 \end{pmatrix} \end{array}$$

Interpretation

On Monday, number of component parts A used is 7, B is 14 and C is 8. in the same way, the number of component parts used for other days can be interpreted.

3.3. The determinant of a square matrix

The determinant of a square matrix A det (A) or |A| is a number associated to that matrix. If the determinant of a matrix is equal to zero, the matrix is called singular matrix otherwise it is called non-singular matrix. The determinant of a non square matrix is not defined.

3.3.1. Determinant of a 2 × 2 matrix

Let A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ = ad – bc

$|A||A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ad – bc

3.3.2. Determinant of a 3 × 3 matrix

A = $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

a(ei – fh) – b(di – gf) + c(dh – eg)

Simplified

3.3.3. Determinant of a 4 × 4 matrix

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

$$|A| = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Simplify 3 ×3 determinants as in ii and then evaluate the 4 x 4 determinants.

3.4. Inverse of a matrix

If for an n x n square matrix A, there is another n x n square matrix B such that their product is the identity of the order n x n, In, that is A × B = B × A = I, then B is said to be the inverse of A. Inverse is generally written as A-1

Hence $AA^{-1} = I$

Note: Only non singular matrices have an inverse and therefore the inverse of a singular matrix is undefined.

3.4.1. General method for finding inverse of a matrix

In order to introduce the rule to calculate the determinant as well as the inverse of a matrix, we should introduce the concept of minor and cofactor.

3.4.1.1. The minor of an element

Given a matrix A = (a_{ij}), the minor of an element a_{ij} in row i and column j (call it m_{ij}), is the value of the determinant formed by deleting row i and column j in matrix A.

Example 3.7

Let matrix A = $\begin{bmatrix} 4 & 2 & 3 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 & 3 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{bmatrix}$

The minors of A are,

$$m_{11} = \begin{vmatrix} 6 & 1 \\ 3 & 0 \end{vmatrix} = 6\times0 - 3\times1 = -3$$

$$m_{12} = \begin{vmatrix} 5 & 1 \\ 2 & 0 \end{vmatrix} = 5\times0 - 1\times2 = -2$$

Similarly

$$m_{13} = \begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix} = 15 - 12 = 3 \qquad m_{21} = \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = 0 - 9 = -9 \qquad m_{22} = \begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 6 = -6 \qquad m_{23} = \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} = 12 - 4 = 8$$

$$m_{31} = \begin{vmatrix} 2 & 3 \\ 6 & 1 \end{vmatrix} = 2 - 18 = -16 \qquad m_{32} = \begin{vmatrix} 4 & 3 \\ 5 & 1 \end{vmatrix} = 4 - 15 = -11 \qquad m_{33} = \begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix} = 24 - 10 = 14$$

3.4.1.2. The cofactor of an element

The cofactor of any element a_{ij} (known as c_{ij}) is the signed minor associated with that element. The sign is not changed if (i+j) is even and it is changed if (i+j)is odd. Thus the sign alternated whether vertically or horizontally, beginning with a plus in the upper left hand corner.

i.e. 3 x 3 signed matrix will have signs $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Hence the cofactor of element a_{11} is m_{11} = -3, cofactor of a_{12} is $-m_{12}$ = +2 the cofactor of element a_{13} is $+m_{13}$ = 3 and so on.

Matrix of cofactors of A $= \begin{pmatrix} -3 & 2 & 3 \\ 9 & -6 & -8 \\ -16 & 11 & 14 \end{pmatrix}$

in general for a matrix M $= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Cofactor of a is written as A, cofactor of b is written as B and so on. Hence matrix of cofactors of M is written as

$= \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$

3.4.1.3. The determinant of a n×n matrix

The determinant of a n×n matrix can be calculated by adding the products of the element in any row (or column) multiplied by their cofactors. If we use the symbol ∆ for determinant.

Then ∆ = aA + bB + cC

or

= dD + eE + fF e.t.c

Note: Usually for calculation purposes we take ∆ = aA + bB + cC

Hence in the example under discussion

∆ = (4 ´ –3) + (2 ´ 2) + (3 ´ 3) = 1

3.4.1.4. The adjoint of a matrix

The adjoint of matrix $\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$ is written as

$$\begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$$

i.e. change rows into columns and columns into rows (transpose of the matrix of cofactors)

The inverse of the matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

is written as $\frac{1}{determinant}$ $\frac{1}{determinant}$ x (adjoint of the matrix)

$$\text{i.e. } A^{-1} = \frac{1}{\Delta} \times \begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$$

Where ∆ = aA + bB + cC

Hence inverse of $\begin{pmatrix} 4 & 2 & 3 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{pmatrix}$

is found as follows

Δ = (4 ´ –3) + (2 ´ 2) + (3 (3) = 1

A = -3	B = 2	C = 3
D = 9	E = -6	F = -8
G = -16	H = 11	I = 14

$$A^{-1} = \frac{1}{1}\begin{pmatrix} -3 & 9 & -16 \\ 2 & -6 & 11 \\ 3 & -8 & 14 \end{pmatrix}$$

(Note: Check if A (A-1 = A^{-1}´ A = 1)

3.5. Solution of simultaneous equations

In order to determine the solutions of simultaneous equations, we may use either of the following 2 methods

i) Matrix inverse method

ii) Cramer's rule

3.5.1. The cofactor method

This method requires that we obtain

a) The minors and cofactors

b) The adjoint of the matrix

c) The inverse of the matrix

d) Premultiply the original by the inverse on both sides of the matrix equation

Example 3.8

Solve the following

$4x_1 + x_2 – 5x_3 = 8$

$-2x_1 + 3x_2 + x_3 = 12$

$3x_1 – x_2 + 4x_3 = 5$

Solution

a) From the system above, we have

$$\underset{A}{\begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix}} \underset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{b}{\begin{pmatrix} 8 \\ 12 \\ 5 \end{pmatrix}}$$

We need to determine the **minors** and the **cofactors** for the above matrix

Definition 3.1

A **minor** is a determinant of a sub matrix obtained when other elements are as shown below.

A cofactor is the product of (-1) i + j and a minor where

i = Ith row i = 1, 2, 3 …….

j = Jth row j = 1, 2, 3 …….

Cofactor of 4 (a11) = (-1) 1+1 $\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = 13$

Cofactor of -2 (a21) = (-1) 2+1 $\begin{vmatrix} 1 & -5 \\ -1 & 4 \end{vmatrix} = 1$

Cofactor of 3 (a31) = (-1) 3+1 $\begin{vmatrix} 1 & -5 \\ 3 & 1 \end{vmatrix} = 16$

Cofactor of 1 (a12) = (-1) 1+2 $\begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} = 11$

Cofactor of 3 (a22) = (-1) 2+2 $\begin{vmatrix} 4 & -5 \\ 3 & 4 \end{vmatrix} = 31$

Cofactor of -1 (a23) = (-1) 2+3 $\begin{vmatrix} 4 & 5 \\ -2 & 1 \end{vmatrix} = 6$

Cofactor of -5 (a13) = (-1) 1+3 $\begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} = -7$

Cofactor of +1 (a23) = (-1) 2+3 $\begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} = 7$

Cofactor of 4 (a33) = (-1) 3+3 $\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} = 14$

The matrix of C of cofactors is

$$\begin{pmatrix} 13 & 11 & -7 \\ 1 & 31 & 7 \\ 16 & 6 & 14 \end{pmatrix}$$

$$CT = \begin{pmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{pmatrix}$$ = Adjoin of the original matrix of coefficients

The original matrix of coefficients

$$= \begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix}$$

Therefore determinant is

$$\begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix} = \begin{array}{ccccc} 4 & 1 & -5 & 4 & 1 \\ -2 & 3 & 1 & -2 & 3 \\ 3 & -1 & 4 & 3 & -1 \end{array}$$

$$= (48 + 3 - 10) - (-45 - 4 - 8)$$

$$= 41 + 57$$

$$= 98$$

The inverse of the matrix of coefficients, will be

$$= \frac{1}{98}\begin{pmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{pmatrix}$$

By multiplying the inverse on both sides of the equation we have,

$$\frac{1}{98}\begin{pmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{pmatrix}\begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \frac{1}{98}\begin{pmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{pmatrix}\begin{pmatrix} 8 \\ 12 \\ 5 \end{pmatrix}$$

$$= \frac{1}{98}\begin{pmatrix} 98 & 0 & 0 \\ 0 & 98 & 0 \\ 0 & 0 & 98 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{98}\begin{pmatrix} 196 \\ 490 \\ 98 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

∴ X1 = 2, X2 = 5, X3 = 1

3.5.2. Cramers Rule in Solving Simultaneous Equations

Consider the following system of two linear simultaneous equations in two variables.

a11 x1 + a12 x2 = b1 ……………(i)

a21 x1 + a22 x2 = b2 ……………(ii)

after solving the equations you obtain

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

and

$$x_2 = \frac{a_{11} b_2 - a_{21} b_1}{a_{11} a_{22} - a_{12} a_{21}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Solutions of x_1 and x_2 obtained this way are said to have been derived using Cramer's rule, practice this method over and over to internalize it. It is advisable for exam situation since it is shorter.

Example 3.9

Solve the following systems of linear simultaneous equations by Cramer's rule:

i) $2x_1 - 5x_2 = 7$

$x_1 + 6x_2 = 9$

ii) $x_1 + 2x_2 + 4x_3 = 4$

$2x_1 + x_3 = 3$

$3x_2 + x_3 = 2$

Solution

i. $2x_1 - 5x_2 = 7$

$x_1 + 6x_2 = 9$

can be expressed in matrix form as

$$\begin{pmatrix} 2 & -5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

and applying cramer's rule

$$x_1 = \frac{\begin{vmatrix} 7 & -5 \\ 9 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 1 & 6 \end{vmatrix}} = \frac{87}{17} = 5\frac{2}{17}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 7 \\ 1 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 1 & 6 \end{vmatrix}} = \frac{11}{17}$$

(ii) can be expressed in matrix form as

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

and by Cramers' rule

$$x_1 = \frac{\begin{vmatrix} 4 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}} = \frac{22}{17}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 3 \\ 0 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}} = \frac{7}{17}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 4 & 4 \\ 2 & 3 & 1 \\ 0 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}} = \frac{9}{17}$$

3.5.3. Solving simultaneous Equations using matrix algebra

i. Solve the equations

2x + 3y = 13

3x + 2y = 12

in matrix format these equations can be written as

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \end{pmatrix}$$

pre multiply both sides by the inverse of the matrix

$$\Delta = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$$

and inverse of the matrix is

$$-\frac{1}{5}\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}$$

Pre multiplication by inverse gives

$$\begin{pmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 13 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Therefore x = 2 y = 3

ii. Solve the equations

4x + 2y + 3z = 4

5x + 6y + 1z = 2

2x + 3y = -1

Solution:

Writing these equations in matrix format, we get

$$\begin{matrix} A & \times & B\overline{X} & = & \overline{b} \end{matrix}$$

$$\begin{pmatrix} 4 & 2 & 3 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

Pre-multiply both sides by the inverse

the inverse of A as found before is $A^{-1} = \begin{pmatrix} -3 & 9 & -16 \\ 2 & -6 & 11 \\ 3 & -8 & 14 \end{pmatrix}$

$$\begin{pmatrix} -3 & 9 & -16 \\ 2 & -6 & 11 \\ 3 & -8 & 14 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 9 & -16 \\ 2 & -6 & 11 \\ 3 & -8 & 14 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 22 \\ -15 \\ -18 \end{pmatrix}$$

hence x = 22 y = -15 z = -18

(Note: under examination conditions it may be advisable to check the solution by substituting the value of x, y, z into any of the three original equations)

CHAP 4.

CALCULUS

4.1. Study objectives

By the end of this chapter, you should be able to:

- know the uses of calculus;
- understand the principles and uses of differentiation;
- be able to calculate the derivatives of common expressions;
- be able to use differentiation to solve typical problems; and
- knowhow to use differentiation to find the maximum and minimum points of various functions.

4.2. Differentiation

Definition 4.1

Differentiation is the process of finding the derivative of a function. It establishes the slope of a function at a particular point. Alternatively, this can be described as establishing the rate of change of the dependent variable (say cost) with respect to an infinitesimally small increment in the value of the independent variable (say activity).

If y = x this is the same as f(x) = x and if y = x^2in function notation f(x) = x^2

Assume now that the independent variable x is altered by a very small amount n what is the rate of change in y caused by the challenge in the value of x?

If x is at same value A, then y is also at the value A. If x is altered by n then its value becomes x = A+ n and as a direct consequence the value of the dependent variable becomes y = A + n, when x gets an increment also y gets a similar increment. It follows therefore that

The rate of change of y with x= $\frac{change\,in\,the\,value\,of\,y}{changm\,in\,value\,of\,x} = \frac{n}{n} = 1\frac{change\,in\,the\,value\,of\,y}{changm\,in\,value\,of\,x} = \frac{n}{n} = 1$

This means that the rate of change (i.e., slope) is constant and equal to one so thaty changes by exactly the same amount as x regardless of the level of activity or the amount of the change. What is the consequence of x changing from **x** = **A to x = A + n** along the function y = x^2

When **x = A,y = A²**

When **x = A + n ,y = {A + n)² = A² + 2nA + n²** ,

Thus, the change in value of **y** caused by the increase of n in the value of **x** is **(A² + 2nA + n²) - A² = A²+ 2nA + n² — A²**, which reduces to $2nA + n^2$ **.**

It follows therefore that: The rate of change of y with **x=** $\frac{change\,in\,the\,value\,of\,y}{changm\,in\,value\,of\,x} = \frac{dnA+n^2}{n} = 2A + n$ $\frac{change\,in\,the\,value\,of\,y}{changm\,in\,value\,of\,x} = \frac{dnA+n^2}{n} = 2A + n$

When the value of the small change ***n*** tends to zero the rate of change becomes ***2A***. This means that at any value on the function $y = x^2$, the rate of change in the value **y** with respect to ***x*** is 2x.This is known as **the derivative** or **differential coefficient.**

Thus, the derivative for the function **y** = x^2 *is 2x*

Note: The small change in the value denoted above as n, is conventionally known as ∂x (delta x). As this value tends towards zero, that is, ∂x → 0 the comparison of the changes in value becomes: Limit $\frac{dy}{dx} = \frac{dy}{dx} =$ ∂ x → 0 Generally written as $\frac{dydy}{dxdx}$, which means the derivative of a function when dx tends towards zero. Thus, for the original function y = x2

$$\frac{dy}{dx} = 2x$$

4.3. Basic rules of differentiation (derivative of function)

4.3.1. Exponential functions

Where the function is= x^2, the derivative $\frac{dydy}{dxdx}$= nx^{n-1}

Example 4.1

i) Given y = x^2 , $\frac{dydy}{dxdx}$ **= 2x**

ii) Given y = $\sqrt[5]{x}\left(\text{ie } x^{\frac{1}{3}}\right)\frac{dy}{dx} = \frac{1}{3}x^{\frac{2}{3}}\sqrt[5]{x}\left(\text{ie } x^{\frac{1}{3}}\right)\frac{dy}{dx} = \frac{1}{3}x^{\frac{2}{3}}$

All that is necessary is to multiply x by its original index and to realise that the new index of x is one less than the original value. When the function contains a constant where y = x^n + c (c is the constant),

$$\frac{dy}{dx} = nx^{n-1}$$

Example 4.2

i) Given y = $3x^1$ find the derivative

ii) Given y = $6x^5$ +17, find the derivative

Solution

i) $\frac{dy}{dx} = 3x^0 = 3\frac{dy}{dx} = 3x^0 = 3$

ii) $\frac{dy}{dx} = 30x^4 + 0 = 30x^4\frac{dy}{dx} = 30x^4 + 0 = 30x^4$

Note:

When you differentiate x to power “1” (i.e., x^1), **x** disappears because anything to power zero is a one, that is, Example 2 i).

When you differentiate a constant it becomes zero. This is so because the derivative measures the rate of change a constant (e.g., fixed cost) by definition does not change. Example 2 ii).

Where the function is a sum, where y = x^n + x^m, $\frac{dy}{dx} = \frac{dy}{dx}$ = = nx^{n-1} + $mx^{m-1}mx^{m-1}$

Example 4.3

Given y = x^3 + $6x^2$, find the derivative.

Solution

$\frac{dy}{dx} = \frac{dy}{dx}$ = $3x^2 + 12x^1$

4.3.2. Chain Rule

Given y = (px + q)n

Let t= px + q and y = t»

Then, $\frac{dy}{dx} = \frac{dy}{dx} =$ p and $\frac{dy}{dx} = \frac{dy}{dx} =$ ntn-1

Therefore $\frac{dy}{dx} = \frac{dy}{dx} =$ = $\frac{dydy}{dxdx}$ x $\frac{dydy}{dxdx}$, this is called the Chain Rule

$\frac{dy}{dx} = \frac{dy}{dx} =$ (p) ntn-1 =pn (px +q)n-1

Example 4.4

Given = (4x2— 7)5 , find the derivative.

Solution:

Let t = 4x2-7 and y= t5

Then $\frac{dtdt}{dxdx}$ = 8x and $\frac{dtdt}{dxdx}$ = 5t4

Therefore $\frac{dy}{dx} = \frac{dy}{dx} = \frac{dy}{dx} = \frac{dy}{dx} =$ x $\frac{dt}{dx} = \frac{dt}{dx} =$ = 5t4(8x)

= $40x^4 = 40x(4x^2 - 7)^4$

4.3.3. Product rule of functions

When the function is a product of two or more functions: it takes the form mn - f(x)g(x). Let **m** and **n** represent functions of **x** and **y** = **mn.** Then

$m\frac{d(mn)}{dx} = m\frac{dn}{dx} + n\frac{dm}{dx}m\frac{d(mn)}{dx} = m\frac{dn}{dx} + n\frac{dm}{dx}$ is called the **product rule**, that is, take **m** as a constant differentiate n then take n as a constant then differentiate **m**.

Example 4. 5

Given y = x2(5x + 4), find the derivative.
Solution:

Let m = x^2 and n = (5x + 4),

$\frac{dmdm}{dx\,dx}$ = 2x and $\frac{dndn}{dxdx}$= 5, therefore $m\frac{dy}{dx} = m\frac{dn}{dx} + n\frac{dm}{dx}m\frac{dy}{dx} = m\frac{dn}{dx} + n\frac{dm}{dx}$
= $x^2x5 + 2x(5x + 4) = 5x^2 + 10x^2 + 8x = 15x^2 + 8x$

4.3.4. Quotient rule

Let **m** and **n** represent functions of **x** and y = $\frac{dydy}{dxdx}$then ,$\frac{dy}{dx} = \frac{n\frac{dm}{dx} - m\frac{dn}{dx}=}{n^2}$, $\frac{dy}{dx} = \frac{n\frac{dm}{dx} - m\frac{dn}{dx}=}{n^2}$, $n \neq 0 \neq 0$ this is called the **quotient rule.**

Example 4.6

Given that $_y$ = $\frac{5x^3+6}{x^4}$, $\frac{5x^3+6}{x^4}$, find the derivative.

Solution

Let m = $5x^3$ + 6 and n = x^4

$\frac{dmdm}{dx\ dx}$= 15$\mathbf{x}^2$ and $\frac{dn}{dx} = 4x^3\frac{dn}{dx} = 4x^3$ therefore $\frac{dy}{dx} = \frac{n\frac{dm}{dx} - m\frac{dn}{dx}=}{n^2}\frac{dy}{dx} = \frac{n\frac{dm}{dx} - m\frac{dn}{dx}=}{n^2}$

$$\frac{\text{dy}}{\text{dx}} = \frac{x^4(15x^2) - 4x^3(5x^3 + 6)}{(x^4)^2}$$

$=\frac{15x^6 - 20x^6 - 16x^3 15x^6 - 20x^6 - 16x^3}{(x^8)\qquad(x^8)}$

$=\frac{-5x^6 - 16x^3 - 5x^6 - 16x^3}{(x^8)\qquad(x^8)} = \frac{-x^3(5x^3+16) - x^3(5x^3+16)}{(x^8)\qquad(x^8)}$

$=\frac{-(5x^3+16) - (5x^3+16)}{(x^5)\qquad(x^5)}$

Example 4.7

Given y =$\frac{5\quad 5}{7x7x^8 7x7x^8}$, find the derivative.

Solution

Take x from the denominator into the numerator, that is, y = $\frac{5}{7}x^{-3}\frac{5}{7}x^{-3}$

Then differentiate (find the derivative)

$$\frac{dy}{dx} = -3x\frac{5}{7}(x^{-4}) = -\frac{15}{7}x^{-4} = -\frac{15}{7x^4}$$

4.4. Finding turning points of functions

The points on the functions where turning points occur gives a maximum or minimum values o*; the function. The second derivative or table method can be used to determine the nature of a turning point.

4.4.1. Use of Second derivative in getting maxima and minima values of functions

If the value of the second derivative is **positive** after substituting the **x** value of the turning point (i.e., by equating the expression of the first derivative to zero and solve for x), then that turning point is a **maxima.** If the value of the second derivative is **negative** after substituting the **x** value of the turning point, then that turning point is a minima.

Procedure to find the maximum or minimum value of a function: • . Given y = **2 x^3+ 6x^2 ,**
Find the coordinates of the maximum and minimum points.

Step 1: Find the first derivative of the function of the given function.

$y = 2x^3 + 6x^2$, $\frac{dydy}{dxdx} = 6x^2 + 12x$

Step 2: Equate $\frac{dydy}{dxdx}$= 0 to find x values where the turning points occur, by solving, **$6x^2 + 12x$** = 0,6x(x+ 2) = 0, x = 0 or x = - 2

Step 3: Obtain values of y by substituting x values in the expression for y.

When x = 0 y = 0, giving one turning point as (0, 0)

When x = - 2, y = 8, giving another turning point as (- 2, 8)

Step 4: Find the second derivative of y by differentiating the first derivative again,

$\frac{d^2}{dx^2}\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx}(6x^2+12x) = 12x+12$

Substitute the x = 0 value in the second derivative, $\frac{d^2yd^2y}{dx^2dx^2}$**= 12x + 12 ,**

When x = 0, $\frac{d^2yd^2y}{dx^2dx^2}$ = 12 gives a positive value, this means that the point (0, 0) is a minimum.

When x = - 2,$\frac{d^2yd^2y}{dx^2dx^2}$ = -12 gives a negative value this means that the point (- 2,8) is a maximum.

Example 4. 8

A function h is defined as $h(x) = x^4 - 2x^2 + 4$.

Required

Determine the turning points and investigate their nature.

Solution:

$h(x) = x^4 - 2x^2 + 4$

i) $h'(x) = 4x^3 - 4x$, $h'(x) = 0$

= > $4x^3 - 4x = 0$

$4x(x^2 -x) = 0$

	Turning points
4x(x + l)(x - 1) = O , x = {0,1.-1}	
h(0) = 0-0 + 4 = 4	(0, 4)
h(-1)= 1-2 + 4 = 3	(-1-3)
h(1) = 1-2 + 4 = 3	(1.3)

ii) **$h'(x) = 4x^3 - 4x$, 4x**

$h''(x) = 12x^2 – 4$

$h''(0) = 0 - 4 < 0 \Rightarrow (0,4)$ is max point

$h''(1) = 12 - 4 > 0 \Rightarrow (1,4)$ is min point.

4.5. Application of differentiation (derivatives) in business calculations

Definition 4.2: In Business and Economics, any manufacturing company or firm incurs costs C in the production and said of the manufactured items or operating a business. These costs consist of two parts i.e. fixed costs and variable costs.

Fixed costs are those costs that remain constant regardless of the number of units produced may include advertisement, rent, product design, utilities, depreciation, set up etc.

Variable costs are those costs which depend on the number of items produced at a certain cost per item.

The cost function is defined as the function of the total cost of producing the number of items (units) say x. It is denoted as C(x).

The cost function C(x) = fixed cost + variable cost

Where variable cost = cost per unit x number of units

Revenue is the total income from the sales of the company or firm. Revenue can also be defined as the amount of money a firm or company brings into the firm or company from the sales. If the company or firm charges a cost of p Rwandan Francs per unit, then the Revenue function can be obtained by multiplying the number of items (units) sold by the price of each item or unit. The Revenue function is denoted as R(x).

Revenue function R(x) = number of items sold x price per item

∴R(x) = px (sometimes the price per item is given as a demand function). The profit a firm or company makes on its products is the difference between the amount of n money it receives from its sales (Revenue) and its cost. The profit function is denoted as P(x) and is obtained using Profit function P(x) = Revenue function - Cost function

∴. P(x) = R(x)-C(x)

Note:

1. If the revenue exceeds the cost, that is. $R(x) > C(x)$, then the firm or company will make a profit.
2. If the revenue and the cost are equal, that is. R(x)'= C(x). then the firm or company will breakeven.
3. If the cost exceeds the revenue, that is. $R(x) < C(x)$, then the firm or company will make a loss.

If the question asks you to find breakeven points, then the following steps are followed:
Either:

first find the cost function and revenue function; then

set the cost function equal to the revenue function and solve; or

find the profit function = R(x) - C(x); then

equate the profit function to zero and solve.

Example 4. 9

A firm produces and sells mobile phones for Frw 150,000 per unit. The fixed costs related to this mobile phone are Frw 30,000,000 per month, while the variable costs are Frw 90,000 per unit.

a) Write down the revenue and cost functions.

b) How many phones must be sold to break even?

Solution:

a) R(x) =number of units sold x price per unit
Let x be the number of units sold
∴ R(x) = 150000x
C(x) = Fixed costs + variable costs
∴ C(x) = 30000000 + 90000x

b) In order to break even, R(x) =C(x)
150000x = 30000000 + 90000x (substituting for R(x) and C(x))
60000x =30000000 (collecting like terms)
∴ x = 500 (dividing through by 60000 on both sides)

∴ 500 phones must be sold in order to break even

Alternative method:

Profit function = R(x) - C(x)

= 150000x - (30000000 + 90000x) (substituting for R(x) and C(x)

P(x) = 60000x- 30000000 (opening the brackets and collecting like terms)

To break even P(x) = 0

60000x - 30000000 = 0

60000x = 3000000
∴ x = 500

Example 4. 10

A firm manufactures radios at a fixed cost of Frw 20000 and the cost of producing a radio is 50x + 4800 in Rwandan Francs where x is the number of radios produced.

Determine:

i) the cost function; and
ii) the selling price if the firm is to break even when it has produced 600 radios.

Solution

i) Cost function = fixed cost + variable cost

= 20000 + (50x + 4800) ***x***

∴C(x) = 20000 + $50x^2$ + 4800x

ii) In order to breakeven:

R(x) = C(x)

=> Number of radios x selling price per radio (S.P) =

20000 + SO^2 + 4800*

=> 600 X S.P = 20000 + 50x2 + 4800x

$$\Rightarrow S.P = \frac{20000+50x^2+4800x \quad 20000+50x^2+4800x}{600 \qquad\qquad 600}$$

$$\therefore S.P = \frac{20000+50x600^2+4800x600 \quad 20000+50x600^2+4800x600}{600 \qquad\qquad 600}$$

$\therefore$S.P = Frw34.833

$\therefore$The selling price would be approximately Frw 35000 per radio.

4.6. Marginal cost function, marginal revenue and marginal profit function

Derivatives of functions are applied in business and economics to:

i) find the marginal cost and marginal revenue at different levels of production; and

ii) find the marginal profit function given information about total cost and total revenue.

4.6.1. Cost function, revenue function and profit function

The marginal cost is the cost incurred in producing the additional unit of a certain commodity given at any level of production. To find the exact (actual) cost of producing a particular item, we use the difference of two successive values of **c(x).** Total revenue of producing **(x +1)** item=**c(x + 1).** Total revenue of producing **x** items =**c(x)**

$\therefore$The exact (actual) revenue of producing the **(x +1)**th item **= c(x + i) - c(x)**

But in this course of study we shall use calculus to determine the marginal cost. Marginal cost can also be defined as the instantaneous rate of change in cost in respect to the duration. For any total cost function C(x) defined by an equation, the marginal cost at any level of production is obtained by finding the first derivative of the cost function.

$\therefore$If C(x) is the total cost function, then Marginal Cost function MC is the derivativeof the cost function i.e. MC = C ‘(x). The example below confirms that the use of calculus is an appropriate method of determining the marginal cost.

Example 4.11

The total cost in million Rwandan Francs incurred per month, Electrical Company for manufacturing x electric irons is given by the cost function

C(x) = 4000 + 5x + 0.04x²

a) Find the actual cost incurred for manufacturing the **501**^{sl} electric flat iron.

b) Find the rate of change of the total cost with respect to when x=500.

Solution

a) Actual cost incurred =c(x + I) - c(x)

= C (501) - C (500)

= (4000 + 5x501 +0.04x5012) - (4000 + 5x500 +0.04x 5002)

= 16545.04- 16500

= 45.04million Frw

b) Rate of change of total cost = Marginal cost = C'(x) = £ = s + o.osx

When **x = 500, Mc** = **5+0.08x500** = **45**million Rwandan Francs

Example 4.12

The cost function of a commodity is given by ***C(x)*** **=** $2x^3 - 5x^2 + 40x + 10$ in dollars.

a) Find the marginal cost at ***x*** = 8

b) Explain what this predicts about the cost of producing one additional unit.

Solution

a) ***C(x)*** **=** $\mathbf{2X^3 - 5x^2 + 40x + 10}$

Marginal cost = C '(x) **=** $\frac{dcdc}{dxdx}$ **=** $\mathbf{6X^2 - 1Ox + 40}$

When **x** = **8,**

Marginal cost = $\mathbf{6x8^2}$ **-1ox8 + 40 = $344**

b) This tells us that if the quantity produced is 8 units, then the total cost of producing one additional unit i.e. 9:h unit will increase by approximately S344.

Example 4.13

REMIX company manufactures CDs and sells them for Frw1,000 each. Given that the cost incurred in the production and sale of the CDs are fixed at $400 plus a variable cost of **2 + 0.001y** in dollars for each **y** number of CDs produced and sold.

a) Write the equations for the total cost, total Revenue and the profit function.

b) Find the marginal cost function.

c) Find the marginal cost of producing y=10CDS and y =60CDs comment on your answers.

Solution

a) Cost function C(y) = fixed cost + variable cost

= 400 + (2 **+ 0.001y) y**

.: C(y)= 400 -2y **+ 0.001y²**in dollars

Revenue function R(y) = number of CDs sold x price per each CD

= ***y x* l000**

=**1000y**

Profit function = Revenue function - cost function

= l000y- (400 + 2y + 0.001y²)

= 998y - 400- 0.001y²

b) Marginal cost function = C '(y) **=** $\frac{dcdc}{dxdx}$ **=** 2 **+ 0.002y** in dollars

c) When **y= 10.** Marginal cost = **2 + 0.002x10 = $2.02** (this means that the quantity produced is 10 CDs, the cost of producing an additional CD will increase by approximately $2.02)when **y = 60,** Marginal cost =**2 + 0.002x60 = $2..12** (this means that the quantity produced is 60 CDs, the cost of producing an additional CD will increase by approximately $2.12)

.: The marginal cost increases as the number of CDs produced increases: this is because the variable cost is increasing.

4.6.2. Marginal revenue function

The marginal Revenue function gives the actual revenue realised from the sale of an additional unit of the commodity given that the sales are already at a certain level. To find the exact (actual) revenue of producing a particular item, we use the difference of two successive values of R(x).

Total Revenue of producing (x +1) item ***=R(x* + 1)**

Total Revenue of producing x items =R(x)

.: The exact (actual) Revenue of producing the (x +1)th item = **R(x + 1) –R(x)**

But in this course of study we shall use calculus to determine the marginal revenue. Marginal Revenue function can also be defined as the instantaneous rate of change in revenue in respect to the duration. However, for any total revenue function R(x) defined by any equation, the Marginal Revenue function denoted as MR at any level of production is obtained by finding the first derivative of the Revenue function.

If R(x) is total revenue function then the marginal revenue function MR is the derivative of the revenue function i.e. MR =R'(x) = $\frac{dRdR}{dxdx}$

Example 4. 15

Given that the total Revenue function for a producing a certain commodity is ***R(x)*** = $80x - 0.20x^2$ in millions of Rwandan Francs and x is in millions of figures produced.

Find the:

a) Actual revenue realised for manufacturing the 61^{st} commodity and

b) Rate of change of the total Revenue function with respect to when ***x*** **=**60 (millions).

Comment on your answer.

Solution:

a) Actual Revenue realised = **R(61) - R(60)**

= **$(80x61 - 0.20 \times 61^2) - (80x60 - 0.20x60^2)$**

= 4135.80-4080

= Frw 55.80 million

d) Rate of change of the total Revenue function = R'(x) = $\frac{dRdR}{dxdx}$

= **80 - 0.40x**

When x =60

R' (60) = 80- 0.40x60 = *sk56* millions

.:Rate of change of the total revenue function at x=60 approximates the actual revenue realised for manufacturing the additional unit. The actual revenue realised and the rate of change of the total revenue function is what we call the marginal revenue function. It is, therefore, better to use the derivative method like the one in b) above instead of the first method in a) above.

Example 4.16

The demand function ***p*** = **2500** – ***5q***in thousands of Rwandan Francs for the manufacturing of q computers demanded per day. The total cost incurred daily in the production of q computers is given by the cost function

C(q) = 0.0001q3 - 4.96q2 + 2000q + 100000 in thousands of Rwandan Francs .

a) Find:

i) revenue function; and

ii) profit function.

b)Find:

i) marginal cost function

ii) marginal revenue function

c) Find:

i) marginal cost when q = 234

ii) marginal revenue when q = 234 and interpret your result

Solution:

i) Revenue function R(q) = number of units produced x price per unit

= q(2500- 5q) = 2500q-5q2

ii)Profit function = Revenue function - cost function

=(2500q- 5q2)- (0.001q3 - 4.96q2 + 2000q + 100000)

P(x) = -0.000lq3 - 0.04q2 + 5OOq- 100000

b) i) Marginal cost function MC = C'(q)= $\frac{dcdc}{dqdq}$= o.ooo3q2-9.92q +2000

ii) Marginal Revenue function MR =R'Cq) = $\frac{dRdR}{dqdq}$= 2500 – l0q

c) i) MC = C'(q)= = 0.0003q2- 9.92q +2000 when q = 234

MC = C'(q)= = 0.0003 x 2342- 9.92x234 + 2000 =-$304.85

This means that the total cost incurred by producing the 235,h computer will decrease by $304.85.

d) ii) ***MR = R'(q)* = 2500- l0q**

When q=234, MR = R'(234) = 2,500 - 10 x 234 = $160. This means that the company will realise an increase of $160 in the total revenue by manufacturing the 235th computer.

4.6.3. Marginal profit function

The marginal profit function is the instantaneous 'ate of change of profit relative to production at a given level of production. The marginal profit function is used to estimate the amount of profit from the next item to be produced.

.: If P(x) is the total profit function, then marginal profit function MP is the derivative of the profit function.

MP = P'(x) = $\frac{dpdp}{dxdx}$

Marginal Profit function can also be obtained using: Marginal profit function = marginal revenue function - marginal cost function

.: P'(x) = R'(x) - C'(x)

Example 4.17

The profit function from the daily sales of a commodity is described by P(x)

= $-0.6x^2$ + 3000x - 300 in thousands of Rwandan Francs .

a) Find the marginal profit function.

b) What is the marginal profit when i) x=2.000 ii) x = 2,500 iii) x=4.000

c) Comment on the rates of change in the profit in b) above.

Solution:

a) Profit function P(x) = **$-0.6x^2 + 3000x - 300$**in thousand Rwandan Francs

.:. Marginal Profit function **P'(x)** = $\frac{dpdp}{dxdx}$= **-1.2x + 3000**

b) i) When x = 2000

The Marginal Profit = - **1.2 x 2000 + 3000 = Frw 600000**

ii) When x =2500

The Marginal Profit = **-1.2 *x* 2500 + 3000 = *Frw* o**

iii) When x = 4000

The Marginal Profit = **-1.2 *x* 4000 + 3000 = *Frw* -1800000**

You will notice that the marginal profit is positive when ***x*= 2000**, this means that producing and selling 2.000 items it gives the maximum profit, when **x= 2500**, no profit is made hence breaking even, when the number of units exceeds 2,500, that

is. When **x= 4000**then the profit decreases hence the marginal profit is negative.

Example 4.18

The price of a product is Frw 5,000 per unit and the total cost function is

C(x) = $5x^2$ + 4000x + 1000 where **x** is the number of units produced and sold in a

specified period of time. Find the marginal profit function.

Solution:

Method 1

Marginal Profit function = P'(x)

We need to first the profit function

Profit 'unction = revenue function - cost function (revenue function = number of units

X price per unit)

P(x) = **$5000x - (5x^2 + 4000x + 1000)$** (substuting for R(x) and C(x))

= **$1000x - 5x^2 - 1000$**

.: P'(X) = $\frac{dpdp}{dxdx}$= **1000-10x**

Method 2

P'(x) = R'(x) - C'(x)

Where R(x) =$\frac{cdcd}{dxdx}$=**5000x** => R' (x)=$\frac{dRdR}{dxdx}$ =**5000** and **C(x) = $5x^2$+4000x + 1000**=>

C'(x) **=** $\frac{dcdc}{dxdx}$**= 10x + 4000**

P'(x) = R'(x) - C'(x)

= 5000 -(10z + 4000)

.: P'(X) = **1000- 10x**

4.7. Function notation and formulae

The ability to use calculus to find the minimum and maximum is very useful in many areas of study. When we minimise our costs and maximise our profits then the business goals can approach the optimum.

The table below shows the basic terms and function notation commonly used to solve some economics problems and make use of the derivative skills:

Term	**Function notation**
Total cost function	C(x)
Marginal cost function	***C'(x)***
Price Function	**p(x)**
Average Cost function	$\frac{C(x)}{x}$
Revenue function	**Rx) =xp (x)**
Marginal Revenue function	**R'(x)=**$\frac{dRdR}{dxdx}$
Profit function	***P(x)*** **= R(x) - C(x)**
Marginal Profit function	**P'(X)= R'(x*) – C'(x)or* p'(x) =** $\frac{dPdP}{dxdx}$

4.8. Minimising average cost

4.8.1. Average cost function

If C(x) is the total cost from the sale of x items, then the average cost function

denoted as $\overline{C}\overline{C}$**(x)** *or* $A_C A_C$*(x)* = $\frac{total\ cost\ C(x)}{number\ of\ items\ or\ units\ x} = \frac{C(X)}{X}\frac{total\ cost\ C(x)}{number\ of\ items\ or\ units\ x} = \frac{C(X)}{X}$

We use the first and second - derivative tests to find the minimum of the average cost function.

Example 4.18

The total cost function for a certain product is c**(x) =** $\frac{5}{3}x^4 - 2x^3 - 3000x^2\frac{5}{3}x^4 - 2x^3 - 3000x^2$in thousand Rwandan Francs of units produced.

a) How many units must be sold in order to obtain the minimum average cost per unit?

b) Find the average cost of the product.

Solution:

a) The average cost per unit $=\frac{total\ cost\ function}{number\ of\ units\ produced}\frac{total\ cost\ function}{number\ of\ units\ produced}$

$=\frac{C(X)C(X)}{X\quad X}$

$$=\frac{\frac{5}{3}x^4 - 25x^3 - 3000x^2}{x}$$

$\overline{CC}(x)=\frac{5}{3}x^3 - 2x^2 - 3000x\frac{5}{3}x^3 - 2x^2 - 3000x$

For minimum average cost per unit, we find the first derivative and equate it to zero,

$\bar{C}\bar{C}(x)= -5x^2+ 50x + 3000$ => $5x^2 -50x -3000 = 0$

=> $5x^2 + l00x- l50x- 3000 = 0$

=>$5x(x + 20) - 150(x + 20) = 0$

=>$(x +20) (5x- 150) = 0$

Either ***x*+ 20 =0 or 5x-150= 0 =>**either ***x* = *-20 or x* = 30**

We reject the negative value **x = - 20** but we must confirm ***x = 30*** will result into a minimum average cost per unit, by taking second derivative

$\bar{C}'\bar{C}'(x)= l0x - 50$

$\bar{C}'\bar{C}'(30) = l0x30 - 50 = 250$ since:

$\bar{C}'\bar{C}'(30) >250$

∴∴ this shows,that x = 30 will obtain the minimum average cost per unit.

b) average cost $\bar{C}'\bar{C}'((x) = \frac{5}{3}x^3 - 25x^2 - 3000x\frac{5}{3}x^3 - 25x^2 - 3000x$

$\bar{C}'\bar{C}'(30) =\frac{55}{33}$-x30^3 - 25 X30^2-3000x30

Example 4.19

Haji and Hajjat Travel Agency wishes to take a group of Muslims to Mecca. The terms are:

i) The number of people per group should be 30 or more;

ii) When the group has exactly 30 people each person is to pay $4000; and

iii) When the number of people exceeds 30, each person's cost is reduced by $40.Use the above information to:

a) Find the number of people who will produce the maximum revenue for the travel agency; and

b) Hence, find the maximum revenue that will be collected.

Solution:

a) For a group of 30 people, total revenue = number of people x cost per person

= 30 x 4,000

= $120,000

Let y be the number of people exceeding 30

→The new cost per person becomes $(4000- 40y)

and the number of people = 30 + y

:. New total revenue = number of people x cost per person

= (30 + y)(4000 - 40y)

R(y) = 120000- 1200y+ 4000y 40y2

R(y) = 120000 + 2800y - 40y2

For maximum revenue R'(y) =0

$$R(y) = \frac{dR}{dy} = 2800 - 80y \Rightarrow \frac{80y}{80} = \frac{2800}{80}$$

$$\Rightarrow y = 35$$

***For maximum revenue .the additional number of people would be* 35.*: The number of people is* 30 + 3 5 = 65**

b) Maximum Revenue R(y) = 120000 - 2800y - 40y^2

= 120000 -2800 x35- 40x35 ^{2} = $169000

Example 4.20

A monopolistic company manufactures and sells x bales of clothes per week. The weekly price demand function is p(x) =1000000 - 1000.x and the weekly cost function *is p(**x**)* **=** 5000000 + 400000**x.**

a) How many bales should be produced to maximise the weekly Revenue?

b) What price should the company charge for the bales?

c) Find the maximum weekly revenue.

Solution:

a) Step 1: find the Revenue function
R(x) = number of units x price function
= x(1000000 - 1000x)
R(X) = 1000000x - 1000x^2

Step 2: find the first derivative of R(x)

R'(X) $\frac{dRdR}{dXdX}$ =1000000- 2000x

Step 3: equate R'(x) =0 and solve for x

1000000-2000 x= 0 $\Rightarrow \frac{1000000}{2000} = \frac{2000x}{2000} \therefore x = 500 \Rightarrow \frac{1000000}{2000} = \frac{2000x}{2000} \therefore x = 500$

Step 4: apply the second derivative test

$\frac{d"Rd"R}{dx^2dx^2}$ = - 2,000 since R"(x) < 0 then it is a maximum

Since x = 500 is the only relative maximum500 bales of clothes should be produced to maximise the daily revenue.

b) Selling price is obtained by substituting for x = 500 in the price function

S.P = p(500) = 1000000- 1000x500 = Frw 500000

c) Total revenue R (500) = 1,000,000 x 500 - 1,000 x 5002 = Frw 250,000,000.

4.8.2. Average Revenue Function

If R(x) is the total revenue from the sales of x items, then the average revenue function denoted as *R(x)*

Or $A_R(x)$ $\frac{Total\,Revenue}{number\ of\ items\ x}\frac{Total\,Revenue}{number\ of\ items\ x}=\frac{R(x)}{x}\frac{R(x)}{x}$

If we want to find the rate at which the average revenue function is changing at Specific value of x we just need to find the derivative of the average revenue function. Maximum average revenue is obtained at the x value where revenue average = marginal revenue:

R(x) = MR = R'(x)

Example 4.21

A firm collected total revenues represented by R(x) = 16000x + 480x2 - 60x3

a) Find the maximum average revenue.

b) Find the value of x which attains maximum average revenue.

Solution:

a) Average Revenue = $\frac{Revenue\ funciton}{number\ of\ item}\frac{Revenue\ funciton}{number\ of\ item}=\frac{168000+480x2-60x3}{x}\frac{168000+480x2-60x3}{x}$

=168000 + 480x - 60x^2

b) $\bar{R}\bar{R}(x)$ = R'(x)

→168000 + 480x - 60x^2 = 480-360x

4.8.3. Maximising profit

We shall use the same techniques of the first and second derivative of the marginal profit in order to maximise profit functions.

Example 4.22

A product can be produced at a total cost c(x) = 800 + 100x^2 + $x^3$where x is the number produced. If the total Revenue is given by R(x) = 60000a - 50x^2

a) Determine the level of production that will maximise the profit.

b) Find the maximum profit.

Solution:

a) Step1: find the profit function:

P(x) = R(x)-C(x) = (60000x- 50X^2)-(800 + 100x^2 + x^3) = 60000x - I50x^2- 800 - x^3

Step 2: differentiate P(x)

P'(x)= 60000- 300x- 3x^2

Step 3: equate to zero and solve 60000- 300x- *3*x^2*=0*

$x2+\ 100x\ -\ 20000\ =\ 0\ x2+\ 100x\ -\ 20000\ =\ 0$ (dividing *through by -3)*

$X^2 + 200x - 100x - 20000 = 0$

$X(x + 200) - 100(x + 200) = 0$

=> Either $(x + 200) = 0\ or\ (x - 100) = 0(x + 200) = 0\ or\ (x - 100) = 0$

=> Either x =-200 or x = 100 x =-200 is discarded because it is negative

Step **4:** apply the second derivative test to prove relative maximum

$P''(x) = -300 - 6x$

Substituting for x =100 $P''(100) = -300 - 6x100 = -900P''(100) = -300 - 6x100 = -900$

Since P"(x) < 0 then x =100 is a relative maximum

The level of production that will maximise the profit is x = 100

b) Maximum profit = P(100) = 60000 xl00 - 150 x 1002- 800 - (100)3

= Frw 3.499,200

Example 4.23

A company made an estimate that q units of its products can be produced weekly at a total cost represented by the function C (q) = 1,350,000 + **3**,000q +30q^3. It also estimated that the total revenue from the sale of q units is **R** (q) = 138,000q

Determine:

a) the profit function;
b) the level of production q that will maximise the profit; and
c) the weekly maximum profit.

Solution:

a) Profit function P (q) = R (q) - C(q) = (138,000q) - (1,350,000 + **3**,000q + 30q^3)
= 135,000q - 1,350,000 - 30q^3

b) Marginal profit = P'(q)= 135,000 - 90q^2 to maximise profit P'(q)= 0 => 135.000 - 90q^2 = 0

$\frac{90q^2}{90}\frac{90q^2}{90} = \frac{135000}{90}$ => $\frac{135000}{90}$ =>1,500 => q = ± 38.7units $q \approx 39q \approx 39$unitsc)Weekly maximum profit = (135,000x 39) - 1,350,000- (30 x 39^2) = Frw 2,135,430

Example 4.24

The total profit function for a product is:

P(x) = 6,000x^3 – 315,000x^2+ 5,400,000x - 300,000 where x is the number of items sold.

a) Find the number of items that will maximise the profit.
b) Hence, find the maximum profit.

Solution:

Stepl: find the marginal profit function P'(x)
P'(x) = 18,000x^2 - 630,000x + 5,400,000
Step 2: equate P'(x) to zero and solve

$P'(x) = 18{,}000x^2 - 630{,}000x + 5{,}400{,}000 = 0$

$\Rightarrow x^2 - 35x + 300 = 0$ (dividing *through by 18000)*

$\Rightarrow x^2 - 15.x — 20x + 300 = 0 \Rightarrow x(x - 15) - 20(x - 15) = 0$

$\Rightarrow (x - 20)(x - 15) = 0$.: either x = 20 or a: = 15

Step 3: Find the second derivative and substitute these two points to determine the relative Maximum

$P''(x) = 36{,}000x - 630{,}000$

P''(20) = 36000 x20 - 630000 = 90000 since P'{20).>0 then at

x = 20 *it is a minima*

P''(15) = 36000 x 15- 630000 = -90000 *since P''*(15) *<0 then at*

x = 15 *it is a maxima.*

c) Maximum profit = P(x) = 60oox³ – 315000x²+5400000x - 300000

P(15) = 6000 *x* 15³ – 315000x I5² + 5400000x15 - 300000

= Frw 30,075,000

4.8.4. Self-test questions

Question 4. 1

The total revenue function of a product is **$R_{(x)} = 5x - 0.01x^2$.**

a) How many units will be produced and sold in order to provide the maximum total revenue?
b) Find the maximum revenue

Solution:

a) Revenue function $R(x) = 5x - 0.01x^2$
Marginal revenue function MR(x) = 5 - 0.02x
For maximum revenue 5- 0.02x = 0

$X = \frac{5 \quad 5}{0.02 \, 0.02}$ = 250 units

b) From $R(x) = 5x - 0.01x^2$
Maximum revenue R (250) = 5(250) - 0.01(250) = Frw625.

Question 4.2

A company manufactures scientific calculators. The price function for each calculator is P(x) = 72 - 0.02x, where x is the number of calculators produced.

a) Find the total revenue function.
b) Find the level of sales that will maximise the total revenue.
c) Hence, find the maximum revenue.

Solution:

a) Total Revenue function, R(x) = xP(x) = x(72 - 0.02x)= 72x - 0.02x2
b) R(x) = 72x -0.02x2
Marginal revenue function MR(x) = 72 - 0.04x

For maximum revenue 72 - 0.04x = 0

$X \frac{72 \quad 72}{0.04 \, 0.04}$ 1,800 sales

a) From $R(x) = 72x - 0.02x^2$
Maximum revenue, R(1.800) = 72(1,800) - 0.02(1,800)= Frw 64,800.

Question 4.3

A commodity produced by a certain firm was sold and the firm collected revenues given by the function R(x) = 140,000 + 900X2–(50x3. Find the largest revenue from the sales of this commodity.

Solution

From revenue R(x) = 140,000 + $900x^2$ - $50x^3$

Marginal revenue$\frac{dR(x)}{dx}\frac{dR(x)}{dx}$=1,800x - $150x^2$

For the largest revenue 1,800x - $150x^2$ = 0, 150 x (12 - x) = 0

Either x = 0 or x = 12

Since $\frac{d^2R(x)}{dx^2}\frac{d^2R(x)}{dx^2}$ = 1,800 - 300x, When x = 0 $\frac{d^2R(x)}{dx^2}\frac{d^2R(x)}{dx^2}$ = 1,800 - 300(0) > 0

And, when x= 12$\frac{d^2R(x)}{dx^2}\frac{d^2R(x)}{dx^2}$=1,800-300(12) < 0

Largest revenue from the sales R(12) = 140,000 + $900(12)^2$ - $50(12)^3$= Frw 183.200.

Question 4.4

A firm found out that its monthly revenue from the sales of its product is related to the money, x in thousand Rwandan Francs spent on advertisements, exhibitions etc. The total revenue *R(x)* = 200000 + 320000x- $8x^2$

a) Find the amount of money that should be spent to maximise their revenue.

b) Find the maximum sales revenue.

Solution:

a) R(x) = 200,000 + 320,000x - 8x2

$\frac{dR(x)}{dx} = 3\frac{dR(x)}{dx} = 3$20,000 - 16x, when 320.000 - 16x = 0

$x = \frac{320000}{16}x = \frac{320000}{16}$ =20,000

Since $\frac{d^2R(x)}{dx^2}\frac{d^2R(x)}{dx^2}$ = -16x<0

Then the amount to spend to maximize revenue is Frw 20,000

b) Maximum sales, R(20.000) = 20,0000 + 320.000(20.000) - 8(20,000)2 = Frw 3,200,200,000

Question 4.5

If the total revenue function for a product is given by R(x) = 128000x - $40x^2$

a) How many units will maximise the total revenue?

b) Find the maximum revenue.

Solution:

a) R(x) = 128.000x-$40x^2$

$\frac{dR(x)}{dx}\frac{dR(x)}{dx}$= 128.000-80x

When 128.000 - 80x = 0. x =$\frac{12800}{80}\frac{12800}{80}$= 1600.

Since $\frac{d^2R(x)}{dx^2}\frac{d^2R(x)}{dx^2}$ = -80 < 0

then 1600 units will maximize the total revenue.

b)Maximum revenue, R(1600)= 128.000(1600) - 40(1600)^2

= Frw 102,400,000.

Question 4.6

A function is given by **y = *x*^{3}- 27*x* + 3.**

Required:

i) Find the values of x at which turning point occur.

ii) Distinguish between maximum and minimum turning points.

Solution:

i) x = - 3. and x = 3

ii) (3, -5.1) minimum, (- 3, 57) maximum

Question 4.7

Given the function ***y* = *(x- 3)*^{2}*(x*** - 2). Prove that $\frac{d^2y}{dx^2}\frac{d^2y}{dx^2}$= 8(3x - 4).

Question 4.8

Given $y = (x+2)^2(x^2-1), Prove\ that\ \frac{dy}{dx} = 2(x+2)(2x^2+2x-1), hence, deternine \frac{d^2y}{dx^2}$

$y = (x+2)^2(x^2-1), Prove\ that\ \frac{dy}{dx} = 2(x+2)(2x^2+2x-1), hence, deternine \frac{d^2y}{dx^2}$

Solution:

$$\frac{d^2y}{dx^2} = 12x^2 + 24x + 2$$

Question 4.9

$$Given\ y = \frac{(x+3)^3}{(4-5x)}, find \frac{dy}{dx}$$

Solution:

$$\frac{dy}{dx} = \frac{(x+3)^2(27-10x)}{(4-5x)^2}$$

Revenue function

Question 4.10

The profit function for a Commodity is described by

P(X) = 192000*- 540*^{2} - *10x*^{3}- 1200000

a) How many units must be sold in order to maximise profit?

b) Find the maximum profit.

Solution*:*

a) 250.

b) 625.

Question 4. 11

The total revenue function for a product is given by R(x) = Frw 144,000x and the total cost function is given by C(x) =Frw (4.500X^2+ 9,000x + 90,000).

a) Find the number of units that should be produced and sold to maximise profit

b) Find the maximum profit.

Solution:

a) Number of units x = 15.

b) Maximum profit = Frw 1,003,500.

Question 4.12

A firm found out that its monthly revenue from the sales of its product is related to the money ,x in thousand Rwandan Francs spent on advertisements, exhibitions etc. The total revenue

R(x) = 200000 + 320000x- $8x^2$

a) Find the amount of money that should be spent to maximise their revenue.

b) Find the maximum sales revenue.

Solution:

a) Frw 20,000

b) Frw 3,200,200

Question 4.13

If the total revenue function for a product is given by R(x) = 128000x - $40x^2$:

a) how many units will maximise the total revenue? and

b) find the maximum revenue.

Solution:

a) 1,600 units.

b) Frw 102,400,000.

Question 4.14

The profit function for a commodity is described by P(x) = Frw (192,000x - $540x^2$ - IOx^3 -

1,200.000).

a) How many units must be sold in order to maximize profit?

b) Find the maximum profit.

Solution:

a) 64 units.

b) Frw 6,254,720.

Question 4.15

The total revenue function for a product is given by R(x) = 144,000x and the total cost function

is given by C(x) =4500x^2 + 9000 x + 90000.

a) Find the number of units that should be produced and sold to maximise profit.

b) Find the maximum profit.

Solution:

a) 15 units.

b) Frw 1,485,000.

Question 4.16

Given the function y = 2x2– 5x + 2, required to:

a) Find the points at which the curve cuts the axes; and

b) Point at which $\frac{dydy}{dxdx}$ = 0.

Solution:

a) (0, 2), (2, 0), ($^1/_2$, 0).

b) (5/4, -9/8)

CHAP 5.

INTRODUCTION TO STATISTICS

5.1. Study objectives

By the end of this chapter, you should be able to:

define the term statistics;
- distinguish between descriptive statistics and inferential statistics;
  - explain the difference between a sample and population; state the limitations of statistics; and
- explain the importance of statistics.

5.2. Meaning of statistics

Statistics is a distinct mathematical science which is the study of the methods that are used in the collection, classification, organization, presentation, analysis and interpretation of data.

For example, if one wanted information about performance in Quantitative Techniques within the last three years, one would seek performance records from the Institute. The sample records available would be large and this would necessitate to arrange performance data records in a grouped frequency distribution table. The data arranged in frequency tables would then be used to ease computations, such as mean, median, variance, etc.

5.3. Importance of statistics

Statistics plays a vital role in most fields of human activity. Statistics plays a very vital role in determining the existing position of unemployment, population growth rate, mortality rate, per capita income etc. in a country. The application of statistics is very wide; it holds a central position in important fields as explained below.

Statistics plays an important role in administration. Statistical data is widely used in making administrative decisions. Every administrator must have sets of program and policies which are formulated in order to meet the targeted objective plans. These plans depend on the correct and sound statistical data. Therefore, statistics is used as a tool for planning. Comparative statistical statements of previous years are used to plan for the coming financial year. For example, if the government needs to revise the salaries of public servants, statistical tools are employed to determine the rise in the cost of living. Another example may include preparation of an organization or company s annual budget is made possible because they mainly use statistics to estimate the expected expenditures and revenue.

Statistics is vital in economics. Statistics is very vital in developing and proving the laws and principles of economics. Knowledge of statistics is very useful in assessing and understanding the economic terminologies and problems, such as economic growth. Inflation rate, population growth rates, unemployment, supply and demand and National Income, that **is,** GDP, GNP, income per capita.

Statistics is essential in business. For the smooth operation of a business, statistical data is very useful. It expresses facts in a definite form and simplifies the complex nature of business. Statistics helps the business person to plan according to market demand, that is, taste of the customers, supply, price, quality of the products, etc., hence making decisions after studying the pattern of events say forecasting sales, expenses, advertising for the products, financial resources, location of the business, to mention but a few.

Therefore, lack of statistical knowledge makes it very difficult to be successful in business.

Statistics plays a central role in banking. The banks use statistics in various ways. The banks apply the principle that all the people, companies, organizations or institutions who deposit their money do not withdraw it at the same time. They use statistical methods based on probability to estimate the number of depositors, how much can be loaned and withdrawn on a certain day. These banks, therefore, make profits out of these deposits by lending them as loans to other people or organizations on either simple or compound interest, depending on the period and the rate.

Statistics plays an important role in accounting. **Accounting is impossible without exactness. Accountants use statistics to provide information to shareholders and other business stakeholders regarding business liability.**

Statistics is essential in auditing. **Auditing uses the sampling techniques to determine whether books of accounts have been prepared and trace sources of errors.**

Statistics is useful in medicine. **Medicine deals with treatments that work often but not always, so treatment success entirely depends on probability. Doctors keep clinical records on a daily basis and, hence, refer to them when a related case reoccurs. In case of an epidemic, appropriate statistical comparisons and clarity of exposition are made**

Statistics used in natural and social sciences. **Such sciences include biology, physics, mathematics, astronomy, chemistry, meteorology, geology, etc. They use statistics to draw conclusions and describe them more precisely, in order to draw conclusions or give the complete description of the original data.**

5.4. Functions of statistics

Statistics can be used for various functions but the most essential include:

- Presenting data in a definite form which makes the statement logical and convincing. Numerous figures can be summarized into a single intelligible figure.
- Statistics reduces the complexity of data. Usually raw data is unintelligible. Ling different statistical measures such as averages, graphs, dispersions, etc.. The unintelligible data can be made simple and can easily be interpreted and conclusions drawn.
- **Facilitating comparison.** Comparison wouldn't be possible without the use of the various statistical measures. Both graphical and numerical measures provide ample scope for comparison.
- **Establishing trends and tendencies.** After studying data over a period of time, a trend can be established which can help in forecasting. Planners can forecast the future produce in agriculture, future population by considering present day figures.
- **Drawing valid conclusions/inferences.** Statistical measures can be used draw conclusions from a given survey which can be used to evaluate different projects.
- **Testing hypotheses.** Statistics can be used to test the truth of new ideas. This helps in developing new theories.

5.5. Limitations of statistics

The methods used in statistics only deal with quantitative attributes of data and leave out the qualitative ones, that is, the methods are not exhaustive. Quantitative attributes include figures and facts. They also answer questions, such as "how much", "how many", while qualitative attributes cannot be expressed numerically and this includes integrity, honesty, intelligence, colour, the statistical methods used cannot bring out these characteristics, hence, a limitation to statistics.

Statistical laws used only apply to the average of a given sample, but when it comes to only one point or an individual from the same sample, the parameter being measured may deviate from the average for the sample. These statistical laws apply to only specific conditions and, therefore, cannot be applied universally. These laws are true on average, for example, one may say that on average all students who sat for a certain paper passed it. Therefore, statistics sometimes gives results on averages and not being 100% reliable, hence, a limitation of statistics.

Some statistical methods used cannot be applied to heterogeneous data, that is the sample is not uniform in all characteristics. For example, one may be interested in collecting data about the characteristics of a certain tribe from a particular community but some of the respondents may be living within the same community but from a different tribe and hence gives incorrect information.

In inference statistics, some errors are encountered in statistical: decisions. Some methods used may not ascertain whether an error was made or not.

Although it is true for the average of the sample, the choice of the sample from the population as a whole may be biased. The information obtained from different samples within the same population may give different results, hence, a limited on of statistics.

The information obtained using statistical data or methods should be interpreted, bearing in mind the characteristics of the sample from which the data was collected In the absence of this knowledge about the sample, the information is liable to misuse, that is wrong interpretations or conclusions may be made. It is, therefore, imperative that the statistics must be used by experts or statisticians only.

Statistical information is not an end in itself; it only provides a means of diagnosing a problem. Other techniques or methods should be applied to use this obtained information to derive a solution or conclusion about the identified problem.

Misunderstanding that a statistical measure can be used as a measure of the accuracy of a measurement. Statistics in general, provides an estimate of the minimal error that might be in the measurement. The actual error can be much greater than the minimal error. Statistics measure the variability of a measurement, not the accuracy of a measurement.

5.6. Branches of statistics

Statistics has two main branches namely:

Descriptive statistics: This is a branch of statistics which deals with methods of collection of data, its presentation and organization in various forms, such as distribution tables, graphs (e.g., ogive, Lorenz curves, etc.), diagrams (e.g., pie charts) and finding measures of central tendency and measures of dispersion or spread which are used in the description of data. Managers, CEOs. etc. make use of descriptive statistics in presenting their annual reports, financial accounts and bank statements.

Descriptive statistics is used to present the data in an understandable way, so that a meaningful description can be made.

Inferential or predictive statistics: This is a branch of statistics which deals with techniques used for analysis of data, making estimates that lead to predictions and drawing conclusions or inferences from limited information taken on sample basis and testing the reliability of the estimates or predictions.

Inferential statistics is used to make comparisons or predictions about a larger group, known as population, using information gathered about a small part of that population called a sample.

Inferential statistics answers questions, such as “what is this data telling us about?” and “what should we do?” Techniques used are forecasting trends, hypothesis testing, kurtosis, skewness, etc.

Worked example:

Distinguish between descriptive statistics and inferential statistics.

Solution:

Descriptive statistics is the branch of statistics that focuses on numerical procedures such as finding the measures of central tendency and dispersion, and graphical procedures, such as ogives, histogram frequency tales, etc., that are used to summaries and process while inferential statistics is that branch of statistics which focuses on using data to make predictions, forecasts and estimates to make better decisions.

Inferential statistics involves making generalizations beyond the given data that descriptive statistics does not do.

5.7. Self-test questions

Question 5.1

What is meant by the **term** statistics?

Solution:

The solution can be found in Chapter 5**,** section 1

Question 5. 2

List the types of statistics.

Solution:

The solution can be found in Chapter 5. section 5.

Question 5.3

Distinguish between descriptive and inferential statistics.

Solution:

The solution can be found in Chapter 5, section 5.

Question 5.4

What are the limitations **of** statistics? **Solution:**

The solution can be found in Chapter 5, section 4.

Question 5.5

What are the functions/uses of statistics?

Solution:

The solution can be found in Chapter 5, section3

STATISTICAL DATA

6.1. Study objectives

By the end of this chapter, you should be able to:

- define the term data and identify types of data:
- define primary and secondary data;
- give examples of primary data and secondary data;
- identify the advantages and disadvantages of primary data;
- give the advantages and disadvantages of secondary data;
- distinguish between qualitative and quantitative data;
- Explain the sources of primary data and list their advantages and disadvantages; and explain the sources of secondary data and list their advantages and disadvantages.

6.2. Data

The word data is technically in plural form, its singular is "datum". Data are numerical facts and figures from which conclusions can be drawn. Such numerical data includes ages of Students in a class, marks obtained in a test, family expenditure; political party a voter intends to vote for, a football team one supports, etc.

6.3. Sample and population

A sample is a group of items selected from a population. A sample is normally selected because the population is too large or costly to study in its entirety. A population is the entire group of all items that are being studied. Examples of populations may include all registered accountants in Rwanda, all financial institutions in the country and all registered voters in the country. Examples of a sample from the population all accountants registered with CPA Rwanda are a sample from all registered accountants in Rwanda , either banks, or micro-finance institutions are samples of financial institutions in Rwanda.

6.4. Types of data

Data may be categorized as discrete and continuous or primary and secondary.

6.4.1. Discrete data

This is the type of data which is in the form derived from countable domain such as 0, 1, 2, 3. Examples of discrete data include the number of television sets sold during the week, marks scored by a student in an exam. Etc.

6.4.2. Continuous data

This is the type of data that may assume any numerical value within a specified interval. Example of continuous data may include the heights of **people,** the weights of objects.

6.4.3. Primary data

This is data directly collected, or observed for the first time, by the researcher for a specific purpose. It is original in nature.

Advantages of primary data

- The data is original.
- The information is obtained from primary data and is unbiased.
- It provides accurate information and is more reliable.
- It gives a provision to the researcher to capture the changes occurring in the course of time.
- It is up to date data, relevant and specific to the required product.

Disadvantages of primary data

- Time consuming to collect.
- It requires skilled researchers in order to be administered.
- It needs a big sample size, in order to be accurate.
- It is difficult to administer because of the large volume of data which is raw.
- It is costlier to collect data using this method and sometimes data
- collected may be bulky and take time to process.

Limitation of primary data: The researcher may have his or her own interpretation of the data, or information gathered.

6.4.4. Methods of getting primary data

Primary data is collected through various methods and these may include:

a) Questionnaire

This is a research instrument consisting of a sequence of questions and other prompts for the purpose of - collecting information from respondents. The questionnaire translates the research objective into specific questions. The answers to these questions provide relevant data for drawing inferences. The questions relate to the problem of inquiry directly or indirectly. The questionnaire should consist of a note briefly explaining the aims and objectives of the inquiry

The researcher should ensure the secrecy of the information as well the details, for example, name of the respondent, if required.

b) Types of questionnaires

There are two types of questionnaires and these include:

- **Self-administered questionnaires.** Examples include mail or posta questionnaires, online questionnaires and delivery and collector questionnaires; and
- **Interviewer administration questionnaire.** Examples include telephone questionnaire and interview schedules.

Advantages or merits of a questionnaire

- They usually have standardized answers that make it simple to compile data.
- They do not require as much effort from the questioner as verbal or telephone surveys.
- Very economical in terms of time, energy and money. Widely used when the scope of inquiry is large.
- Data collected by this method is not affected by the personal bias of the researcher.

Disadvantages or demerits of a questionnaire

- They are sharply limited by the fact that the respondents must be able to read the questions and respond to them.
- Questioner may have a problem in question construction and wording those respondents may give varying answers.
- The respondents may not be cooperative and honest hence affecting the accuracy of the information.

c) Guidelines for questionnaire construction

- Use of clear and comprehensive wording which are understandable for all educational levels
- Use of correct grammar in wording.

- Use of statements that will be interpreted in the same way by respondents from different sample areas of the same population.
- Assumptions about the respondents should not be made Questions should be impersonal and non-aggressive.
- Use of positive statements, hence avoiding negative ones.
- Items that contain more than one question per item should be avoided, for example, do you give your children fruits and meat?
- Use of only one aspect of the construct you are interested in per item.
- Anticipate of receiving open ended answers from the respondents. Therefore, the questions should be multiple choices, simple alternative and open ended.

In the simple alternative questions, the respondent chooses between alternatives such **as** "yes" **or** "no"," true' **or** "false" while in the open ended questions the respondents are given **maximum** freedom in answering the questions. **For** example, what are the causes of corruption? In the multiple-choice questions, the respondent chooses the best alternative **from** the ones that are given.

- Use **of** statements where respondents that have different opinions or traits will give different responses

6.**4.5. Statistical survey**

This is **a** method for collecting quantitative information about items in a population.

Advantages of statistical survey

- Accommodates large sample sizes.
- Response scales are the same hence making it easy to compile data.
- It is simple.

Disadvantages of statistical survey

- Misinterpretation of data results.
- Inappropriate use of data analysis procedures.

Direct personal observations.

The data is collected by direct person interviews. With this method, the researcher directly contacts the respondents, solicits for their cooperation and enumerates the data.

Advantages or merits of direct personal observation

- The method is simple to administer.
- With this method both the researcher and the respondent are present at the time of collecting data.
- This method provides most accurate information as the researcher collects the data personally.
- Suitable for all types of respondents, whether educated or not. The data recorded is accurate.

Disadvantages or demerits of direct personal observation

- The data may be inaccurate and inefficient in case the researcher is inexperienced, biased not good natured. Short tempered and dishonest.
- It is quite costly.
- It is time consuming.
- It is only effective when the scope of inquiry is small

6.4.6. Interview

This is involves asking individuals the required information. There are two types of research interviews:

- One-to-one interview this may be either face-to-face interviews or telephone interviews.
- One-to-many interviews.

Disadvantage of interviewing

- Inaccurate or false data may be given to the interviewer. This may be due to:

i) misunderstanding the question,

ii) forgetfulness or

iii) deliberate intent to mislead.

- If a number of interviewers are employed, they may record the answers in the same way as the investigator himself would.

6.4.7. Secondary data

This is processed data that have been already collected and readily available from other sources. In order for the secondary data to be reliable the following requirements must be satisfied.

- **Relevance of the data.** The data should satisfy the requirements of the problem under investigations, that is, concepts used must be the same and the data should not be outdated, units of measurement must be the same.

- **Accuracy of the data.** In order to ensure how accurate, the data is, the following requirements must be considered:

i) specifications and methodology used;
ii) margin of error should be examined; and
iii) dependability of the source must be seen.

- **Availability of the data.** It has to be seen that the kind of data under investigation is available or not. In case the data is not available then in this case one uses the primary data.
- **Sufficiency of the data.** Adequate data must be available.

Advantages of secondary data

- It is economical; it saves expenses and efforts, since it is obtainable from other sources.
- It is time saving, since it is more quickly obtainable than the primary data. It provides a basis for comparison for the data that is collected by the researcher.
- It helps to make the collection of primary data to be more specific, since, with the help of secondary data, one is able to identify the gaps and inefficiencies, so that the additional or missing information may be collected.

Disadvantages of secondary data

- Accuracy of secondary data is not known.
- Data from which the secondary data is obtained may be outdated.
- Secondary data may not fit in the framework of the research factors, for example, units used in the secondary data collection
- Users of such data may not have as thorough an understanding of the background as the original investigator, and so may be unaware of such limitations.

6.4.8. Sources of secondary data

Secondary data may be obtained from:

- Government publications. National institute of Statistics Rwanda collects, compiles, organises and publishes statistical data on a regular basis. Some of these important publications include trade journals, population growth, birth and death rates, etc.
- Government autonomous bodies, such as REB, RRA, central Bank, BRD, REMA, etc., collect and display statistical data on examination results, taxation, inflation rate, development and environment, respectively.
- International publications. International agencies publish regular reports of international

importance. These agencies include the International Monetary Fund (IMF), International Labour Organization (ILO), World Meteorology Organisation, etc.

- Private publications. Some private sectors also publish reports, for example, Rwanda Exchanges Board , NGOs, etc.
- Newspapers and magazines. Various newspapers, as well as magazines, have statistical information on social and economic aspects. Some of these include New Vision, Monitor, Imvaho, Newtimes, Olupiny, Red Pepper, The Observer, Bridal Magazine, etc.
- Archives. Libraries have historical data of significance.
- Internet websites.

6.4.9. Differences between primary and secondary data

Primary data	Secondary data
Data is original and, thus, more accurate and reliable.	Data is not reliable. -
Extra precautionary measures needed not be put into consideration.	Extra precautionary measures must be put into consideration.
Collecting and gathering data is expensive.	Collection and gathering of data is cheap.
Data is not easily accessible.	Data is easily accessible through published books, internet and many other sources.
Collection and gathering of data requires a lot of time.	
Data gives detailed information.	Data may not be adequate.
Most of the data is homogeneous.	The data is not homogenous.

6.4.10. Qualitative and quantitative data

Quantitative data

Quantitative data is information about quantities, that is, information that can be measured and written down with numbers. Some examples of quantitative data are your height, your shoe size and the length of your fingernails.

Qualitative data

Qualitative data is information about qualities, information that can't actually be measured. Some examples of qualitative data are the softness of your skin, the grace with which you run, and the colour of your eyes.

Qualitative versus quantitative data

Qualitative Data	**Quantitative Data**
• Deals with descriptions. • Data can be observed but not measured. • Colours, textures, smells, tastes, appearance, beauty, etc. • Qualitative —► Quality	• Deals with numbers. • Data which can be measured. • Length, height, area, volume, weight, speed, time, temperature, humidity, sound levels, cost, members, ages, etc. • Quantitative — Quantity

Example New student in a School Qualitative Data • Friendly classmates. • Civic minded. • Environmentalists. • Positive school spirit.	Example New student in a School Quantitative Data . 700 students. . 280 girls. 420 boys. •55% on bursary scheme. •270 students taking mathematics.

6.4.11. Variable and attribute

Attribute is a characteristic of an object (person, thing, etc.). Attributes are closely related to variables. **Variable** is a **logical** set of attributes. Variables can "vary", for example, be high or low How high, or how low, is determined by the value of the attribute in, that is, an attribute could be just the word "low" or "high".

While an **attribute** is often intuitive, the **variable** is the **operationalised** way in which the attribute is represented for further **data processing.** In data processing, data are often represented by a combination of items (objects organised in rows), and multiple variables (organised in columns).

6.4.12. Response errors

Response errors represent a lack of accuracy in responses to questions**.** They can be attributed to different factors, including a questionnaire that requires improvements, misinterpretation of questions by interviewers or respondents and errors in respondents' statements.

Types of errors

Recall errors that occur when a respondent forgets expenditures made during the period covered by the survey (which corresponds to the calendar year) or when a respondent provides an erroneous value because of the time interval that has elapsed between the time of purchase and the date of the interview. Recall errors are probably the survey's largest source of response error, since the reference period is long (12 months) and a wide range of information is requested.

Note: To reduce the magnitude of this type of error, the respondent is also encouraged to consult various documents (bills, bank statements, etc.) so as to provide more accurate data.

Telescopic error which consists of including in the reference period events that occurred before or after it. The use of the calendar year is considered to provide a good marker for the start of the reference period. Furthermore, since the reference period is a long one, telescopic error has less impact.

Responses by proxy can also contribute to response error. The household member who made expenditure is generally best able to report it accurately. This is definitely the case with, say, personal purchases. Expenditures reported by an intermediary are more likely to be tainted by response error, and this type of error tends to have a greater effect on certain types of expenditures.

Among other sources of response error, the extent of the respondent's cooperation should not be overlooked. For personal reasons, the respondent may choose not to mention particular expenditures or decide to twist the facts.

6.5. Sample frame

In statistics, a sampling frame is the source material or device from which a sample is drawn. It is a list of all those within a population who can be sampled, and may include individuals, households or institutions.

In many practical situations, the frame is a matter of choice to the survey planner, and sometimes a critical one.

a) Qualities of sample frame

An ideal sampling frame will have the following qualities:

- All units have a logical, numerical identifier.

- All units can be found -their contact information, map location, other relevant information is present.
- The frame is organized in a logical, systematic fashion.
- The frame has additional information about the units that allow the use of more advanced sampling frames.
- Every element of the population of interest is present in the frame.
- Every element of the population is present only once in the frame.
- No elements from outside the population of interest are present in the frame
- The data is 'up-to-date'.

b) **Sample frames problems**

- Some members of the population are may not be included in the frame.
- The non-members of the population may be included in the frame.
- Duplicate entries. A member of the population is surveyed more than once.
- Groups or clusters.

Self-test questions

Question 6.1
What is primary data?
Solution:
The solution can be found in Chapter 6, section 3c.
Question 6.2
Mention and give a brief explanation on the methods used to collect primary data.
Solution:The solution can be found in Chapter 6, section 4.
Question 6.3
Write down the merits and demerits of primary data.
Solution:The solution can be found in Chapter 6, section 4c).
Question 6.4
What is meant by secondary data?
Solution:The solution can be found in Chapter 6, section 8.
Question 6.5
Briefly explain any six sources of secondary data.
Solution:The solution can be found in Chapter 6, section 9.
Question 6.6
What are the advantages and disadvantages of secondary data?
Solution: The solution can be found in Chapter 6, section 10.
Question 6.7
Distinguish between primary and secondary data.
Solution: The solution can be found in Chapter 6, section 10.
Question 6.8
What are the requirements that must be satisfied in order to collect secondary data?
Solution:The solution can be found in Chapter 6, section 8.
Question 6.9
Write down the guidelines that must be followed in order for one to construct a good questionnaire.
Solution: The solution can be found in Chapter 6, section 4d).
Question 6.10 What is a sample?
Solution: The solution can be found in Chapter 6 , section 2
Question 6.11
What is a population?
Solution: The solution can be found in Chapter 6, section 2

CHAP 7.

SAMPLING TECHNIQUES

7.1. Study objectives

By the end of this chapter, you should be able to:

- identify the various sampling techniques and give their advantages and disadvantages;
- describe sampling techniques commonly used in business and commerce are divided into three major categories, namely random sampling, quasi-sampling and non-random sampling;
- describe sampling techniques commonly used in business and commerce; and
- Classify sampling techniques in different categories.

7.2. Random sampling

Random sampling is also referred to as probability sampling. In this method, each item of the population under consideration has an equal chance of being selected as part of the sample. Each item (subject) in the population is selected independent of the rest of the items.

Random sampling is further broken down into simple random sampling and stratified sampling.

7.2.1. Simple random sampling

Simple random sampling technique is the most basic method among the probability sampling techniques. This method uses a sampling frame that lists all members of the target population, giving each member an equal chance of being selected in the sample.

This method can be applied in the following cases:

- When RBS is testing the efficiency of fuel pumps in fuel stations, they may sample 25% of the fuel stations in one town.
- When an Auditor is sampling 10% of the company's invoices for completeness and compatibility with total yearly turnover.

i) Advantages of simple random sampling

- The sample is generated with ease.
- It eliminates bias because every member of the population has equal chance of being considered.
- The sample is always representative of the population.

ii) **Disadvantages of simple random sampling**

- The need for population listing.
- Time consuming, since the entire population is involved.
- There is a tendency to down play or over emphasize certain significant attribute of the population.
- Quite costly when the population covers a wide geographical area.

7.2.2. Stratified sampling

This is a sampling technique applied to a heterogeneous (having groups with varied attributes) population that can be subdivided into homogeneous (with manufactures television sets, radios and cameras). These are homogeneous groups referred to as strata; hence, the name stratified sampling.

Stratified sampling, therefore, involves obtaining simple samples from each of the strata of the population and the simple samples combined to give the stratified sample of the population. This ensures that each homogeneous segment of the population is proportionally represented in the sample.

Generating a stratified sample involves the following steps:

- Identification of the population strata.
- Calculation of the proportion of each homogeneous segment.
- Splitting the population sample into the proportions above.
- Obtaining corresponding simple samples from the strata.
- Combining the various simple samples to give the stratified sample.

Note: Stratification of the population depends on the situation under consideration.

This method is applicable when there is need to:

- highlight a particular subgroup within the population, ensuring the presence of the key subgroup, for example, first time young mothers on a weekly antenatal visit to a health unit who are at risk among the expectant mothers; and
- observe the relationship between subgroups, for instance, the common attributes shared by the female and the male importers in the Rwanda n business community

a) Advantages of stratified sampling

- It eliminates bias.
- Ensures a better coverage of the population.
- Easy to apply and achieves better precision than the simple random sampling

b) Disadvantages of stratified sampling

- It requires an extensive sampling frame
- Strata of importance may be selected subjectively.
- Time consuming.
- Quite costly.

7.2.3. Quasi-random sampling

Quasi-random means 'nearly' or 'almost' random. This technique is not truly random sampling but it is as representative as random sampling in certain conditions. This is appropriate when random sampling is:

- not possible; and
- too expensive to carry out.

There are two quasi-random sampling techniques namely, systematic sampling **and** multi-stage sampling.

7.2.3.1. Systematic sampling

This method is used when the population is listed in a given order or some of it is physically in evidence, for example, a row of houses or clients visiting a particular webpage in one hour on a given day.

The sample is drawn according to some predetermined point (place/object) chosen at random, and then systematically sample the nth item in the population, the number n chosen depending on the size of the sample required.

For a population of 100 items, if a sample of five items is required the n becomes 20, that is, the 20^{th}, 40^{th}, 60^{th}, 80^{th} and 100^{th} items form the sample.

The technique is particularly applicable for a homogeneous population. For instance, the invoices of a bookshop for one financial year are **a** homogeneous population for an auditor, provided their value or relationship to the bookshop is of no consequence to the investigation.

a) **Advantages of systematic sampling**
- It is easy to implement.
- Works where there is no sampling frame as long as the items are physically in evidence.
- Saves time and is not costly.

b) Disadvantages of systematic sampling
- Bias may occur where recurring sets in a population are possible.
- This sampling technique is not perfectly random. Once the starting point has been determined, all the subjects are predetermined, hence, the name quasi-random sampling.

7.2.3.2. Multi-stage sampling

This method is most suitable when the population is spread over a relatively wide geographical location. It can be summarized in the following steps:
- Splitting the area into regions.
- Randomly selecting the sample from the regions.
- Randomly selecting the sample from the major towns.
- Splitting the samples towns into zones.
- A sample is then selected from the zones.

The process can go on until the desired final sample is achieved using either of the methods described above, depending in the availability of the sampling frame or not.

a) Advantages of multi-stage sampling
- Takes less time to accomplish.
- Requires a small man power to execute.
- Less costly compared to random sampling.

b) Disadvantages of multi-stage sampling
- It is subject to bias, especially when a few regions are selected.
- It is not truly random. At each sampling stage, the population left out has no chance of ever making it to the final sample.

5) Selection of random samples

There are two methods which are commonly used namely lottery method and table of random numbers method.

a) Lottery method

This is a very popular method for selecting random samples. It involves naming c numbering all items of a population on separate slips of paper of identical size, colour and shape. The slips are then folded and mixed up in a container. A blindfold selection of slips to make up the sample is made, mixing the slips in the container' before each picking is done.

This method can be employed to select 20 advertising firms from 100 top advertising firms in E. Africa to participate in advertising promotion of the regions tourist potential. The names or numbers representing the 100 firms are written of identical paper, a raffle draw conducted to pick the 20.

Note: It should be noted that as the population becomes larger the use of the lottery method also becomes challenging.

b) Table of random numbers

Random sampling can also be done by use of random sampling numbers. These consist of ten digit from 0 to 9, obtained by chance (randomly) by use of computer

A number of random tables have been developed, among which are:

a) Toppett's table of random number;
b) Fisher and Yates numbers; and
c) Kendall and Babington Smith numbers.

These tables eliminate bias when a sample is obtained from a given sampling frame.

7.3. Non-random sampling

Non-random sampling is a method used when the random sampling techniques above are impossible or not practical to use. This method comprises of cluster and quota sampling techniques.

7.3.1. Cluster sampling

Cluster sampling is applicable when:

- The sampling frame does not exist; and
- The population under study is distributed over a large geographical area.

This method is implemented by selecting one or more geographical area, sampling all the members of the target population that can be identified.

For instance, in a survey on profitable small enterprises (SME'S) in central region in Rwanda , first one or two major urban areas (say , Kigali and Butare) could be chosen and at least two not so much urbanized areas (say , Muhanga and Rubavu) could also be considered . Each SME is identified and an external auditor asker to look at their gooks of accounts to establish those that made profit in the previous fiscal year. 0131862515

Advantages of cluster sampling

- It is a good alternative to multi-stage sampling.
- Little organization or structure needed in the selection of the subjects.
- It is cheap to carry out.

Disadvantages of cluster sampling

- Being non-random may lead to selection bias.

7.3.2. Quota sampling

This method employs a team of interviews each assigned a set number of subject. A lot of responsibility is placed on the interviewers because selection of subjects is left to them entirely. The interviewers should be therefore well trained and should have a responsible professional attitude.

This method is commonly used in market research.

a) Advantages of quota sampling

- The population is usually stratified.
- It has no non-responses errors.
- Less costly and convenient.

b) Disadvantages of quota sampling

- Sampling is non-random and may, therefore, lead to selection bias.
- Quite challenging to implement and may lead to interview bias.

7.4. Non-Probability Sampling

7.4.1. Purpose sampling

Also, it is called a deliberate or judgment sampling. In this when the researcher deliberately selects certain units for study from the universe is known as purpose sampling.

Steps:

1. Certain units are deliberately selected on judgment of researcher and nothing is left.
2. Units selected must be representative of the universe.

Merits:

1. Under proper safeguard, it is economical and time saving.
2. In this, knowledge of composition of universe, ensure the proper representation of a cross-section of various strata.
3. It is useful when certain units are important to be included to fulfill the requirement of investigation.
4. It is practicable, when randomization is not possible.

Demerits:

1. Considerable prior knowledge of population is necessary which in most cases is not possible.
2. If the controlling safeguard is not affective, it is possibility of bias selection.
3. Calculation of sample error is not possible, so the hypothesis framed cannot be tested.

7.4.2. Quota sampling

It is a special type of stratified sampling. In this method, the population is stratified on some basis, preferably on the characteristics of population under study. After this the number of sample units to be selected from each stratum is decided by the researcher in advance. This number is known as Quota which may be fixed is fixed according to specific characteristics such as income group, sex, occupation, etc. the investigator usually apply their judgment in choice of sample and try to complete quota assigned form each stratum.

Merits:

1. It is a combination of stratified and purposive sampling, thus it make best use of stratification economically.
2. Proper control brings more accuracy in results.
3. It is useful when sample frame is available.

Demerits:

1. Control over field work is difficult because of personal bias, hence affecting results.
2. Since it is not based on random sampling, the sample error as well as standard error cannot be estimated.
3. May not be true representative, its not randomly selected.
4. Substitution of strata can affect results.

7.4.3. Convenience Sampling

It is known as unsystematic, careless, accidental or opportunistic sampling. Under this sample is selected according to convenience of the investigator. This may have the base of availability of data, accessibility of units, etc.

It can be used when

1. universe is not defined
2. Sample is not clear
3. A complete source list is not available.

7.5. Self-test questions

Question 7.1

Briefly distinguish between the quota and block (cluster) methods of selecting a sample.

Solution:

The solution can be found in Chapter 7, section 7 and section 3a)

Question 7.2

Distinguish between random sampling and non-random sampling, giving their merits and Demerits.

Solution:

The solution can be found in Chapter 7, sections 1 and 6.

Question 7. 3

Write short notes on the following:

a) Stratified sampling.
b) Multi-stage sampling.

Solution:

a. The solution can be found in Chapter 7, section 2.
b. The solution can be found in Chapter 7, section 4.

Question 7.4

a) List any **one** example of the following sampling techniques:

i. Random sampling.
ii. Quasi random sampling.
iii. Non-random sampling.

b) Describe each of the sampling techniques listed in a) above.

Solution:

The solutions to these questions can be found in Chapter 2, sections 3, 5 and 6

METHODS OF DATA PRESENTATION

8.1. Study objectives

By the end of this chapter, you should be able to:

- construct various tables for a data set;
- interpret the data presented in tables;
- identify advantages of each type of table; and
- Identify limitations of each type of table.
- Represent data on pictograms;
- represent data on simple and compound bar charts; and
- Illustrate data on line graphs, histograms and frequency polygon and curves.
- **Whenever data is presented pictorially or graphically it gives a better visual impression**

Statisticians must organise, explore and summarise data for easy interpretation. This chapter shows how data is presented in tabular form for various purposes.

8.2. Tables

A table is a layout of information or data in form of columns and rows. They are the simplest form of data display. Some of the possible reasons for which a table may be constructed are to:

- Present the original figures in an orderly manner;
- show a distinct pattern in the figures;
- summarizes the figures for easy interpretation; and
- publish salient figures which other people may use in the future statistical studies.

8.2.1. Principles of table design

The principles to construct a table so that it achieves its objective in the best way possible are as follows:

- Simplicity. A table with too much details or which is too complex is harder to understand. It is better to show only a little that is understood than to show all having nothing understood.
- Comprehensive explanatory title indicated above the body of the table.
- The source must be stated. All figures come from somewhere and a statement of
- The source must be indicated below the table.
- Units of measurement must be clearly stated. Where big figures are involved, it is possible to reduce their number by indicating in the title or in the headings the number of thousands, or other multiples of ten each figure represents.
- Double counting of figures should be avoided because it creates misleading presentation.
- The headings to columns and rows should be unambiguous. Use short headings to remove ambiguity.
- Figures should be approximated before tabulation to reduce unnecessary details in the table. The number of decimal places retained should be uniform.
- The arrangement of data in a table may be chronological, alphabetical or according to

magnitude to facilitate comparison.

- Totals should be shown where appropriate. They are used in a table for one of the following purposes:
    - To give the overall total
    - Indicate that preceding figures are sub-divisions of the total.
    - To indicate that all items have been accounted for.

8.2.2. Types of tables

8.2.2.1. Frequency tables

Cases where one variable appears more than one times, **frequency distribution tables** are used. These tables show the items and the number of times each item appears.

A frequency distribution table can be constructed running horizontally as in Example 2 below, or vertically as in Example 3 below. The data is arranged in order of size either increasing or decreasing.

Example 8.1

In a certain Institute, tutors are paid an hourly rate and the amount depends on their experience. The table shows how many tutors are paid each rate.

Hourly rate (Frw)	8,000	10.000	15,000	20,000
Number of tutors(frequency)	4	12	7	2

Example 8.2

The workers in a certain company were appraised and given either of the ratings: good, very good, fair, and extremely good. The table shows the number of workers in each rating.

Example 8.3

In order to find the number of people in each household, a sample of 18 households were surveyed. The results were:

2 2 1 2 4 3 3 2 3
1 1 3 2 3 6 3 4 5

Rating	Number of workers (frequency)
Fair	5
Good	14
Very good	8
Extremely good	**3**
Total	**30**

Required:

a) **Classified frequency tables**

Formed by classifying/grouping data values into classes. The table will show the frequency of data values within each category.

Household size (number of people)	Tally	Number of households (frequency f)
1	///////	3
2	/////	5

3	//////	6
4	//	2
5	/	1
6	/	1
Total		If=18

Prepare a frequency table.

Example 8.4
The table shows a total of 35 accountants distributed according to the age brackets.

Class	Frequency
30-34	6
35-39	8
40-44	12
45-49	9
Total	35

Example 8.5
The frequency distribution table shows the rate for buying United States dollars at foreign exchange bureau in Kigali. This is a table that summarizes data that have been classified on two dimensions or scales (row and column). The data can be viewed by focusing on the time pattern or by comparing the variables.

8.2.2.2. Two-way cross classification (contingency) table

Example 8.6

i) The table shows establishments in six major sectors from 2008 to 2009

ii) The table shows establishments in six major sectors from 2008 to 2009

	Establishments		
Industry	**2008**	**2009**	**Total**
Construction	19	9	28
Health	28	22	50
Manufacturing	160	175	335
Agriculture	8	9	17
Education	352	199	551
Hotels	88	77	165
Total	**655**	**491**	**1146**

Source: Rwanda Bureau of Statistics

a) **Advantages of tables**
- Easy to prepare.
- The table enables required figures to be located more quickly.
- It enables comparisons between different levels to be made more easily.
- Reveals patterns within the figures which cannot be seen in the narrative form.

b) **Disadvantages of presenting data in tables**
- Figures may be double represented for continuous data.
- Interpretation may be difficult especially where big figures have been reduced.

- Involve computations before tabulation that may be hard.

c) Limitations of table construction

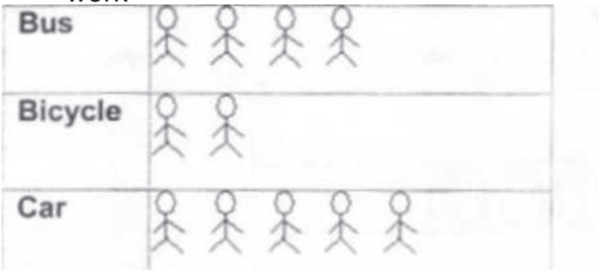
- Extraction of information from narrative form for presentation may be hard.
- Lay out of information especially what should be put in a row or column.
- Not all data can be tabulated.

8. 3. Pictograms (Pictograph)

A pictogram is a pictorial representation of data. These are simple diagrams that use pictures or symbols. Each picture or symbol may represent one or more units of the data.

Example 8.7

The following pictograph shows the number of people using the various types of transport to go to work



This implies that there are eight people who use Bicycles and there are no people who walk to work.

The choice of picture depends on the type of data being described.ie when one is dealing with data involving humans, bottles, etc. one might choose to use symbol pictures of humans as in the above example. Note, in all pictograms there should be a representative scale.

a) Method of construction

- Identify the data to be represented and choose a suitable representative symbol. In addition, you need to assess why you have to represent it and to whom.
- Make drawings of your representative symbol. In some cases, making a drawing of your symbol is very simple. In other cases, representing data which is not of the same magnitude is much more complex (i.e., if one symbol of a car represents 100 cars, it becomes hard to represent 83 cars. So, be mindful of what is the most appropriate symbol to represent the data).
- Modify the drawing of the symbol to remove any features, details or characteristics that aren't essential to its communicative effectiveness.

Example 8.8

The pictograph below shows the number of canned drinks sold in three different shops in a week.

Shop A	
Shop B	
Shop C	

Represents 20 cans

a) What is the total profit of shop A, if the profit gained on each drink is 500 Rwandan Francs? Walking

b) If the total number of cans sold is 180 how many symbols must be drawn for shop C?

c) What is the difference between the number of cans sold by shop B and the number of cans sold by shop C?

Solution:

a) Total profit of shop A= 20 * 4 x 500 = 40,000/=
b) Nine symbols must be drawn for shop C. (9 x 20 = 180)
c) Difference between shop B and shop C = 20 x 2 = 40 cans

Advantage of pictograms

- Easy to understand and interpret.

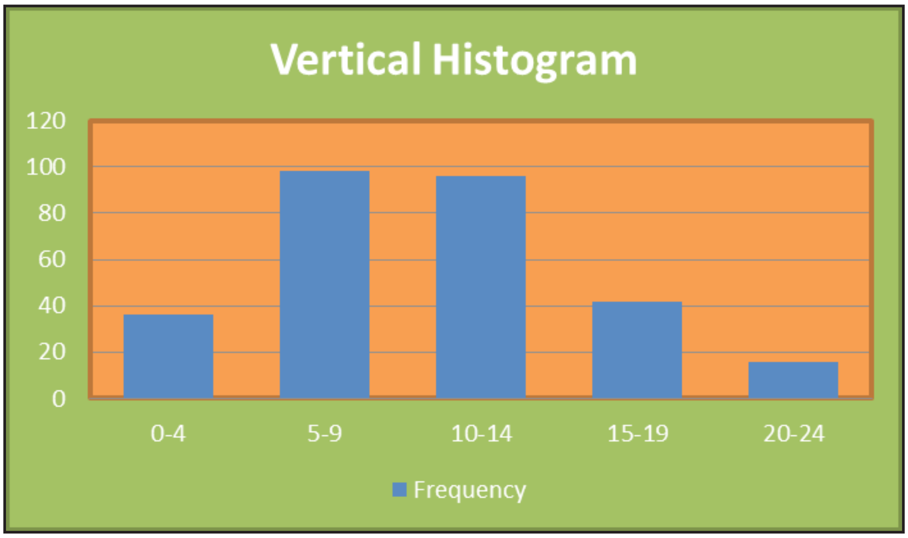
Disadvantages of pictograms

- Can be troublesome to construct if complex symbols are used.
- Not accurate enough for serious statistical presentation.
- Magnification of symbols (using areas or volumes) can be confusing unless the values of figures being represented are clearly shown.
- Complicated to represent data by fractional symbols or pictures.
- Maintaining the consistency of the size of the symbols to use is not easy.

8.4. Charts

Data in bar charts is represented by a series of parallel bars of equal width. The length of each bar is proportional to the frequency of the data it represents. The bars should be of the same width and the gaps between adjacent bars should be equal. Bars may be drawn horizontally or vertically but all must start from the same base line.

Vertical drawing



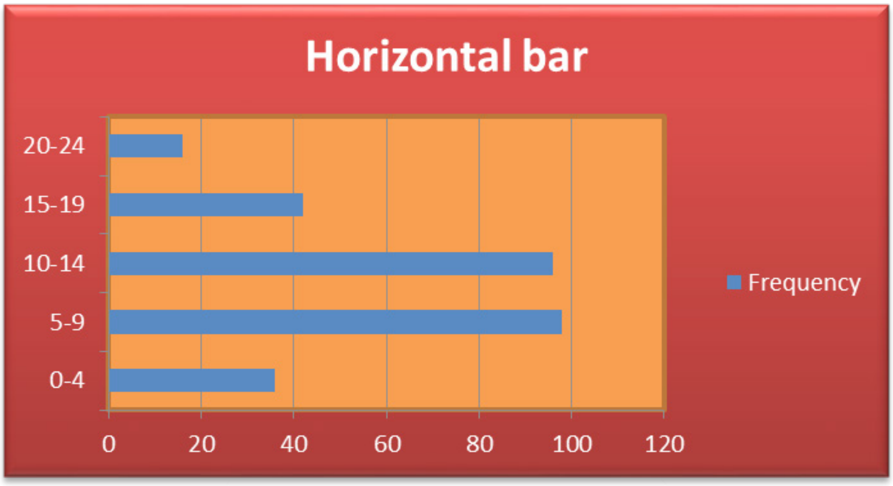
Example 8. 9

classes	0-4	5-9	10-14	15-19	20-24
Frequency	36	98	96	42	16

Example 8.10

If drawn vertically, the diagram is often called a *column bar graph* as shown in the example above.

Horizontal drawing



If drawn horizontally, the diagram is often called a *horizontal bar graph* as show-in the example below.
We have two types of bar charts:

- The simple bar charts.
- Compound bar charts.

Steps for drawing bar charts

- Make the chart slightly wider than it is high.
- Start the scale values with zero and write them in 5's or 10's or integral multiples of 5's and 10s.
- Make the width of the space between the bars less than the width of the bars and leave spaces between the margins of the chart and the first and last bars.
- Shade or cross hatches the bars.

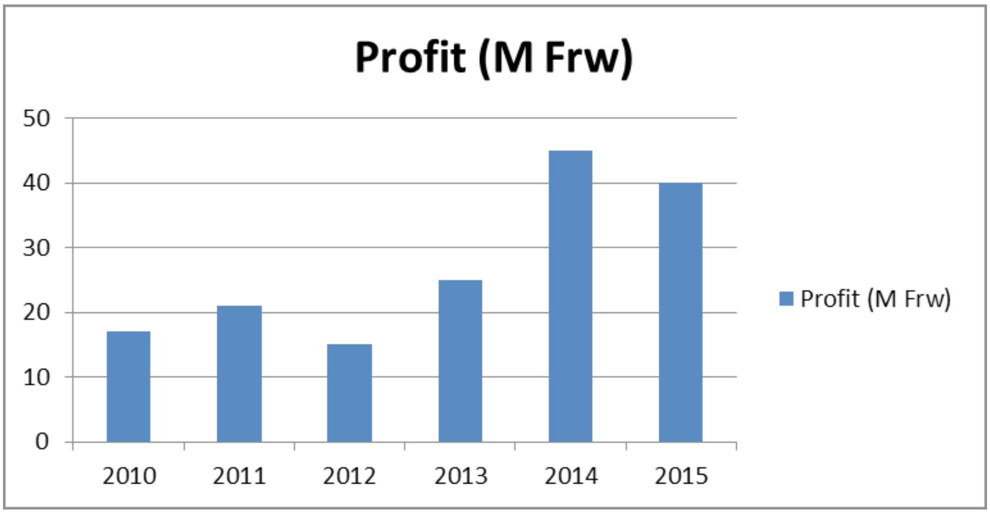
8.5. Simple bar charts

In a simple bar chart, a separate bar for each class is drawn to a height proportional to the class frequency. You can identify the figure that each bar represents at the base of the bar or you can use a key to show that a color or shade indicates a particular item. One of the most important points to remember when drawing Simple bar charts is that you must start the scale from zero.

Example 8.11

Draw simple bar diagram to represent the profits of a bank for six years

Years	2010	2011	2012	2013	2014	2015
Profit (M Frw)	17	21	15	18	45	40

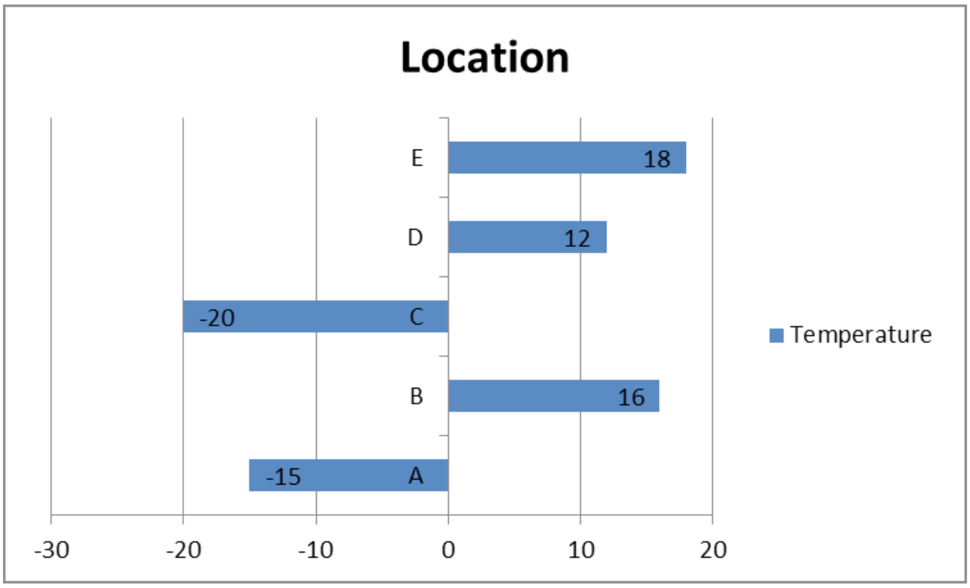


Simple bar chart showing the profit of a bank for six years

Features:

They can be drawn with vertical or horizontal bars, but must show a scaled frequency axis.They are easily adapted to take account of both positive and negative values, for example the seasonal temperatures of cities in the world using the bar chart in the example below:

Location	A	B	C	D	E
Temperature	-15	16	-20	12	18



Example 8. 12

• Two bar charts can be placed back to back for comparison purposes. For example, the performance of males and females in ICPAR EXAMS over a period of time.

Example 8.13

The table below shows earning in a district by sex in thousands of Rwandan Francs.

Salary X in (Frw 000)	100-300	301 - 500	501 - 700	701 - 1000	1000 - 300C
Males	6	5	9	4	3
Females	9	6	5	2	1

Required: Draw back to back bar charts showing the salary earning
Solution:

Advantage of simple bar charts

Easy to understand the values being represented by bars (since a scaled axis is always present). Note that there are no significant disadvantages.

8.6. Compound bar charts

A compound bar chart is useful when you want to compare two or more quantities on one chart. If you are comparing many different quantities, it may be useful to colour code the different bars.

Steps for drawing a compound bar chart

- Identify the data you wish to present on a compound bar graph.
- Present your data in a table for easy reference. In this example, you would assign a column to each of the cities and then assign a horizontal row for each of the three years, then insert the relevant data into each corresponding cell.

Example 8. 14
For example, hypothetical data collected on crime rates in cities A, B and K in 2009, 2010 and 2011, respectively, would be ideal for producing this kind of graph.

	A	B	K
2009	70	55	85
2010	60	75	80
2011	65	70	73

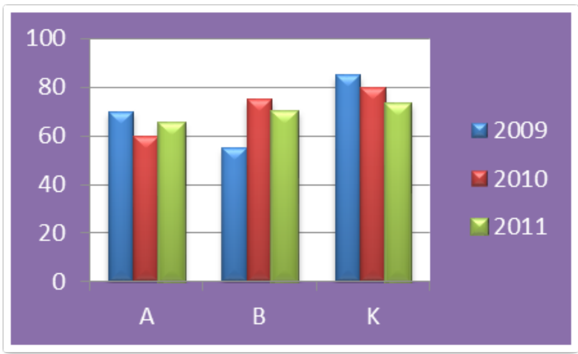
Solution:

	A	B	K
2009	70	55	85
2010	60	75	80
2011	65	70	73

- example, if your range of data values are all below 10 then a scale of 1:1 will suffice; if they are up to 1,000, use a scale of 1:100. Along the horizontal axis, make your data parameters with a main heading and then several subheadings. In the example from step 1, the names of cities would be the main headings and the years would be the subheadings.

- Add your data to the graph. Continuing the example, if A 2009 is the first piece cf data on the graph, plot this information from the table onto the chart by drawing a solid bar. The bar's width will be the width of the subheading and its height wi correspond to the value of the data. Repeat this for each of the subheadings for A then move onto B and K until all the data from the table is plotted onto the chart.

The final bar for the above example will appear as below:



8.7. Pie charts

Pie charts are a way **of** representing data **so** that it **is** easier to understand and interpret. A pie chart shows the totality of the data being represented using sectors of circle. The circle is split into sectors, each one being drawn in proportion to the data it represents. A whole circle is 360 degrees (360") so the sectors of a pie chart will be fractions of the 360°.

The formula to determine the angle of a sector is Angle of sector= $\frac{frequency\ of\ data}{Total\ frequency} * 360$

$= \frac{frequency\ of\ data}{Total\ frequency} * 360$

a) Steps to construct a pie chart

- Calculate the angle of each sector using the above formula.
- Draw a circle using a pair of campuses.
- Use a protractor to draw the angle for each sector.
- Label the circle graph and all its sectors.

Example 8.15

In an institute, there are 750 students in level one, 420 students in level two and 630 students in level three.

Required:

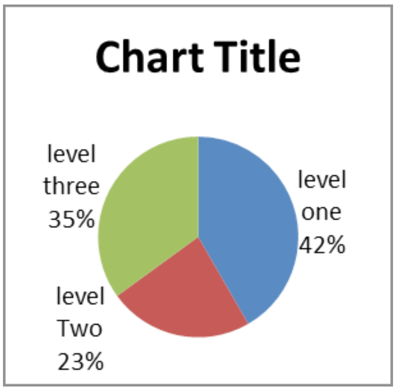
Draw a pie chart to represent students in each level.

Solution:

Total number of students = 750 + 420 + 630 = 1,800. level or size of angle = $\frac{750}{1800}x360^0 = 150^0$, $\frac{750}{1800}x360^0 = 150^0$,

level two; size of angle = $\frac{420}{1800}x360^0 = 840^0 \frac{420}{1800}x360^0 = 840^0$

level three size of angle = $\frac{630}{1800}x360^0 = 126^0 \frac{630}{1800}x360^0 = 126^0$
Draw the circle, measure the angle in each sector. Label each sector and the pie
Chart Number of students per level



a) Advantages of pie charts
- Easy to interpret.
- Good for comparing data in relative terms.
- Easy to show the percentage of total for each category.

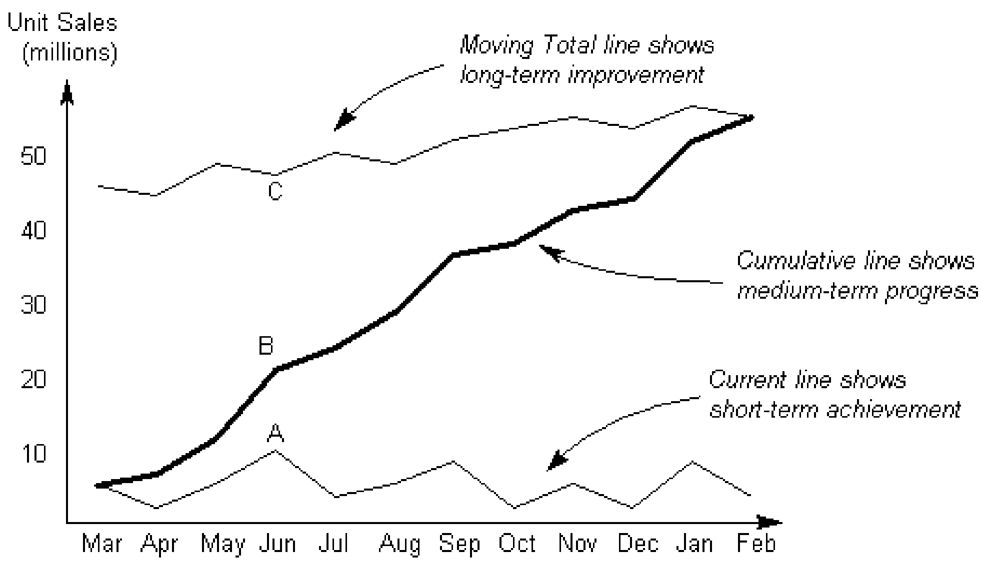
b) Disadvantages of pie charts
- Computation of angles can be tedious if fractions are involved.
- Can be untidy or over crowded if there are many sectors (i.e., nine or more) and different shadings or colouring are being used.
- Totals are unknown unless specified.

8.8 . Z-Chart

There are three in one line graphs representing actual figures, cumulative figures and moving totals in a Z-chart. The Z-charts are normally used to portray monthly time series over a single year.

Example 8.16

The figure bellow shows the Z-chart



a) **Interpretation of the above graph**

- Over the past 12 months, sales have been up and down, ranging for 2 million Rwandan Francs to 12 million Rwandan Francs per month. Across the months, there '-as beer no visible trend.
- Looking at the longer-term rolling 12 month figures, there is what semis to be a visible upward trend.
- The medium-term line shows how sales have built up over the past 12 months
- The line diagram describes the actual time series values. Normally the I me series consists of monthly measurements over one complete year that is January through December.
- The cumulative curve describes the accumulated time series values for monthly measurements, the cumulative curve and the line graph all start from the same point. The second plot will be January plus February, and so on. The more removed from a straight line it is the more variation there has been in the actual monthly figure.
- The moving total line graph is drawn such that each point describes the current month's figure plus the previous eleven month's figure to form a twelve-month total. Collectively the twelve values so obtained are called moving totals.

b) Steps on how to draw a Z-chart

- Use available data on a given activity. This may be the sales per month, words written by an author per week or calls handled by a customer response centre per day, etc.
- Identify the major review period to be considered. If your minor review period is one day, then the major review period is more likely to be one month or three months.
- For each minor review period (three months in the examples here), build the longer-term review by totaling the data for the first three months (nine moving totals in one year).
- Draw the Z-chart, as in the diagram above.

Example 8.17

The table below shows number of visitors to a hospital in two consecutive years from January to December.

	Monthly visitors (2010)	Monthly visitors (2011)
Jan	81	90
Feb	338	370
Mar	1042	900
April	878	680
May	781	690
Jun	777	590
Jul	612	699
Aug	753	790
Sep	888	900
Oct	786	650

Nov	771	680
Dec	528	500

Required:

i) Find the moving totals.
ii) Plot the Z-chart.

Solution:

i)	Monthly Visitors(2010)	Monthly Visitors(2011)	Cumulative Total(2011) T	Moving Total(2011)
Jan	81	90	90	8325
Feb	338	370	460	8447
Mar	1042	900	1360	8305
Apr	878	680	2040	8107
May	781	690	2730	8016
Jun	777	590	3320	7829
Jul	612	699	4019	7916
Aug	753	790	4809	7953
Sept	888	900	5709	7965
Oct	786	650	6359	7829
Nov	771	680	7039	7738
Dec	528	500	7539	7310

Step 1: Obtain the cumulative total for 2010, that is, 8235.

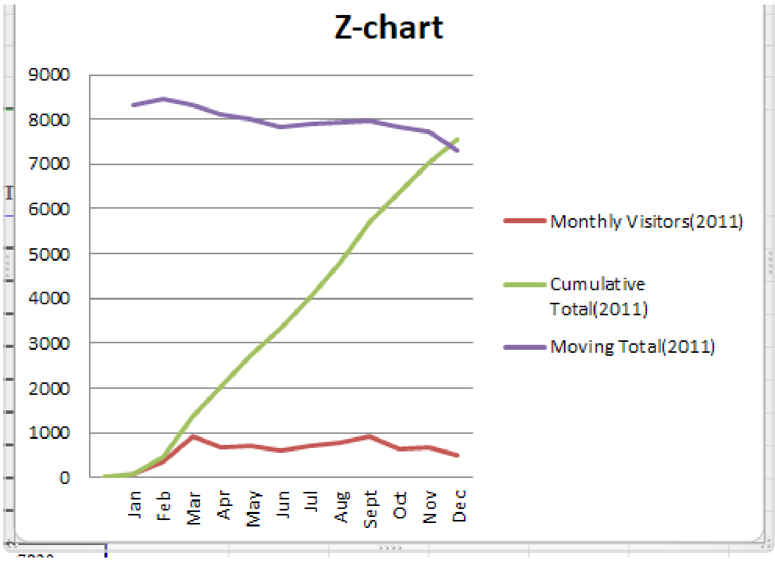
Step 2: Find the cumulative totals for 2011 as shown in column 4 in the table above.

Add the 1st cumulative total of 2011 to the cumulative total 2010 to get the 1st moving total, that is, 8,235 + 90 = 8,325.

Step 3: Subtract the number of visitors for February in 2010 from the 1s' moving total and add the number of visitors for February in 2011 to get the 2nd moving total, that is. (8,325-338) + 370 = 8,447.

Step 4: Subtract the number of visitors for March in 2010 from the 2nd moving total and add the number of visitors for March in 2011 to get the 3rdmoving total, that is, (8,447 - 1,042) + 900 = 8,305

Step 5: Repeat Step 5 to obtain the remaining moving totals as shown in the table above.



8.9. Line graphs

A line graph represents the data concerning one variable on the horizontal and other variable on the vertical axis. It uses points and lines to show change over time.

It is plotted from a set of points and then joined by a line. Different data sets can be plotted on the same graph but a key must be used to identify each data set.

a) Steps for constructing a line graph

Use the data provided to choose an appropriate scale. All scales start an origin. Draw and label the scale on the horizontal axis and vertical axis. Plot the points on the graph. Join the points with line segments. Write the title of the line graph.

The monthly visitors of hospital in 2017 are

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sept	Oct	Nov	Dec
Monthly Visitors(2017)	81	338	1042	878	781	777	612	753	888	786	771	528

Plot Line graph

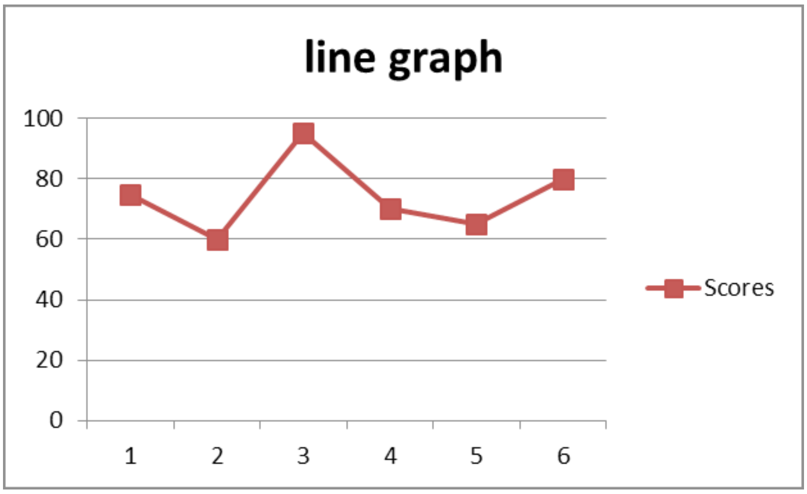
Example 8.18

The data below shows James' BM test grades for six sets. Draw a line graph**.**

Test

Scores
1
2
3
4
5
75
60
95
70
65

6 80
Jams' test scores



a) Advantages of line graphs

- Easy to construct.
- Easy to understand
- A sense of continuity is given by a line diagram which is not present in a bar chart.
- The perfect medium to enable direct comparisons.
- Can be used for prediction.

a) Disadvantages of line graphs

- Might be confusing if too many diagrams with closely associated values are Compared together.
- Where several diagrams are displayed, there is no provision for total figures.

8.10. Histogram

A histogram is a graphical representation of a grouped frequency distribution. It is a graph, including vertical bars, with no space between them. The class-intervals / boundaries are plotted along the horizontal axis and the respective class frequencies on the vertical axis using suitable scales on each axis. For each class, a bar is drawn with base as width of the class and height as the class frequency.

Example 8.19

The data below the weights of 35 students in a school

Classes	45<50	50<55	55 < 60 60 < 65	70 < 75
Frequency	5	9	125	4

Draw a histogram to represent the data

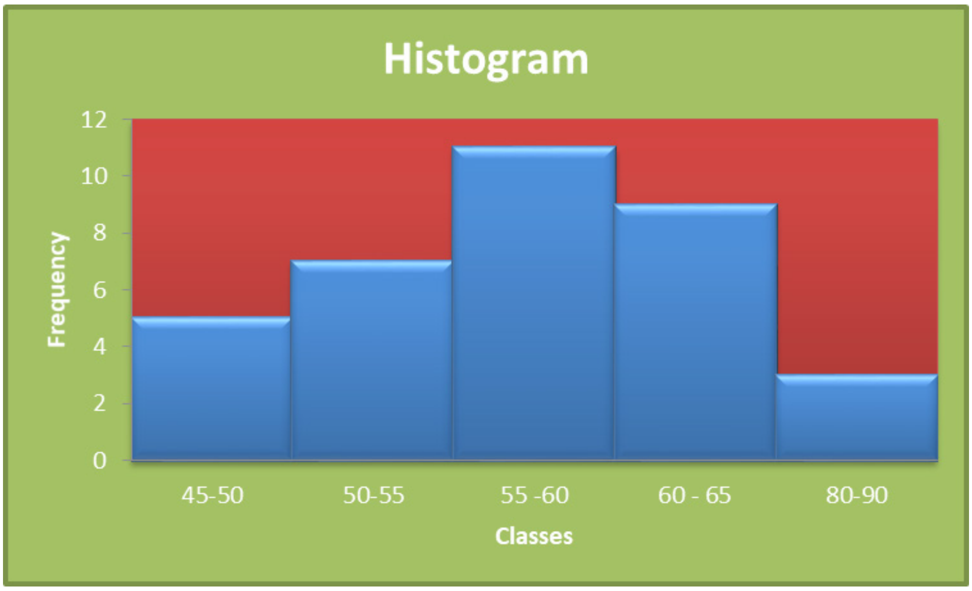
Classes	45-50	50-55	55 -60	60 - 65	70 - 75	80-90
Frequency	5	7	11	9	6	3

Solution: Step 1: On a paper, draw two perpendicular lines and label the horizontal and vertical axes as class boundaries and frequency respectively.

Step 2: Along the horizontal axis, take classes of equal width: 45-50. 50-55 ...As the axis starts from 45-50, take first interval 40-45 before it and put a kink (zig zag) on axis before that.

Step 3: Choose a suitable scale on the vertical axis to represent the frequency.

Step 4: Draw the bars as shown below.

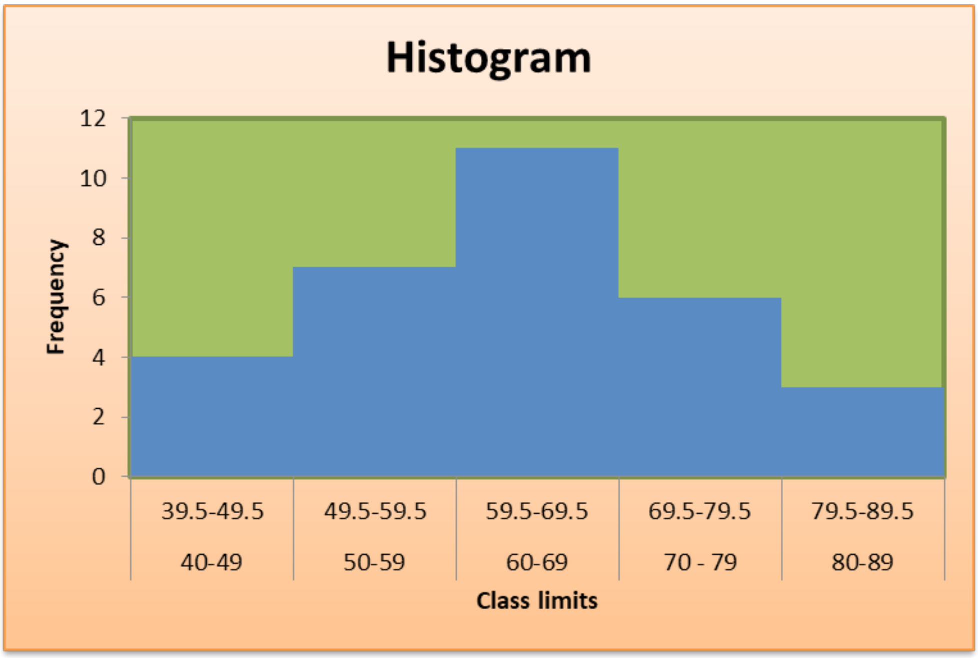


Example 8.20

The data below scores of 31 students in a test

Classes	40-49	50-59	60-69	70 - 79	80-89
Frequency	4	7	11	6	3

Classes	Class limits	Frequency
40-49	39.5-49.5	4
50-59	49.5-59.5	7
60-69	59.5-69.5	11
70 - 79	69.5-79.5	6
80-89	79.5-89.5	3



Required:
Note: In this case, class limits must be found first (i.e., subtract 0.5 from 40 and add 0.5 to 49 to give 39.5 - 49.5 as a class limit). Do the same to the other classes. Follow the steps in example one to draw a histogram.

8.11. Frequency polygons

A frequency polygon is the graph joining of the mid-points of the tops of the adjoining bars. The mid-points of the first and the last classes are joined to the mid-points of the classes preceding the first and succeeding the last respectively at zero frequency to complete the polygon.

Example 8.21
Draw a frequency polygon.

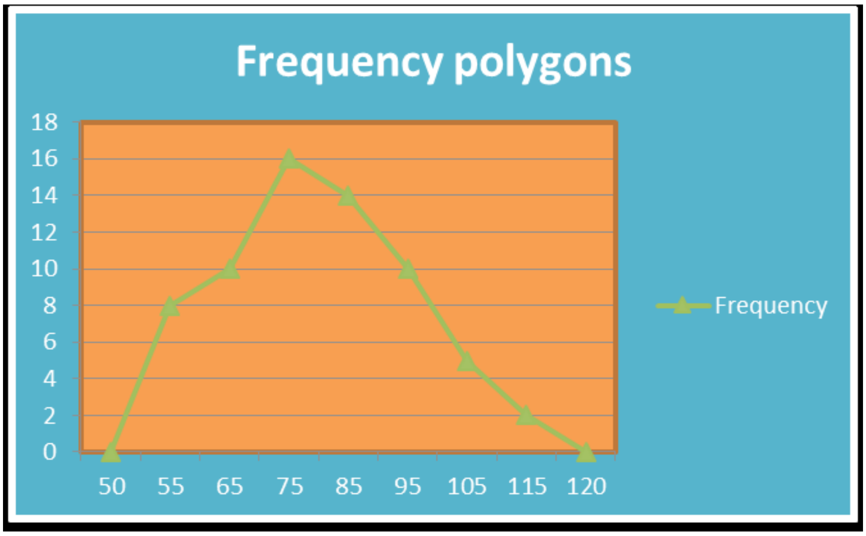
Solution:
To draw a frequency polygon without drawing a histogram we go through the following steps:
Step 1: Find the class-marks (mid points) of different classes. They are 55, 65,75,85,95,105,115.
Step 2: Draw two perpendicular lines to represent the class mark on the horizontal and Frequency on the vertical.
Step 3: Plot the ordered pairs (frequency against class mark) (55,8), (65,10). (75.16), (85,14), (95.10), (105,5) and (115,2).
Step 4: Join the points and complete the polygon as explained before.

The weights of 65 children are given by the following table:

Weight	50-60	60-70	70 - 80	80 - 90	90-100	100-110	110-120
Frequency	8	10	16	14	10	5	2

Required:

Weight	Mid Class	Frequency
50-60	55	8
60-70	65	10
70 – 80	75	16
80 – 90	85	14
90-100	95	10
100-110	105	5
110-120	115	2



Frequency polygons can always be used in place of histogram, but are particularly useful:

- when there are many classes in the distribution; or
- if two or more frequency distributions need to be compared

8.12. Curves

8.12.1. Ogive (cumulative frequency curve)

An Ogive is drawn with the cumulative frequency plotted against the upper boundaries of the relevant interval. This kind of curve allows you to read off numbers below (or less than) a specified value.

To draw a cumulative frequency, curve the points are joined with a smooth curve. These curves can be used for estimation purposes.

Example 8.22

The frequency table below shows marks of candidates in an exam

Marks	**10-20**	**20-30**	**30-40**	**40-50**	**50-60**	**60-70**	**70-80**	**80-90**
Frequency	**2**	**8**	**12**	**18**	**28**	**22**	**6**	**4**

Required:

Draw a 'less than'Ogive curve for the following data.

Steps to draw an Ogive:

Obtain the cumulative frequencies

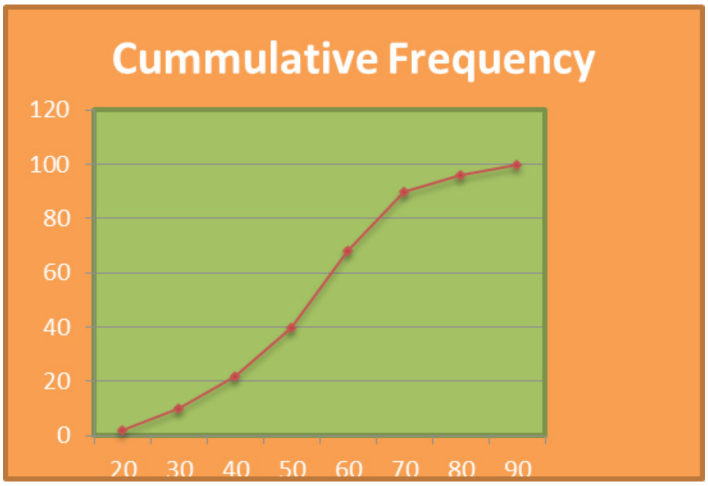
Draw and label the horizontal to represent class boundaries and vertical to represent the cumulative frequencies.

Plot the points having the actual upper boundaries and the cumulative frequencies,(20,2), (30,10), (40,22), (50,40), (60,68), (70,90), (80,96) and (90,100)

Join the points plotted by a smooth curve.

The curve should be joined to a point on the horizontal axis representing the actual lower boundary of the first class.(i.e. 10,0)

Scale: horizontal axis 1cm=10marks, vertical axis 1cm = 10 cumulative frequency.



8.12.2. Lorenz curves

A Lorenz curve is composed of a cumulative percentage frequency curve of the less than type. A Lorenz curve plots cumulative percentage frequency against cumulative percentage class totals.

A class total for any frequency distribution class is the total value of all items belonging to the class. Class totals are estimated using:

Class total=class mid-point x class frequency

A Lorenz curve has a horizontal axis that begins at 0% and ends at 100% and a vertical axis that begins at 0% and ends at 100%.The purpose of such a curve is to demonstrate relative proportions of values represented along the horizontal versus values represented along the vertical axes.

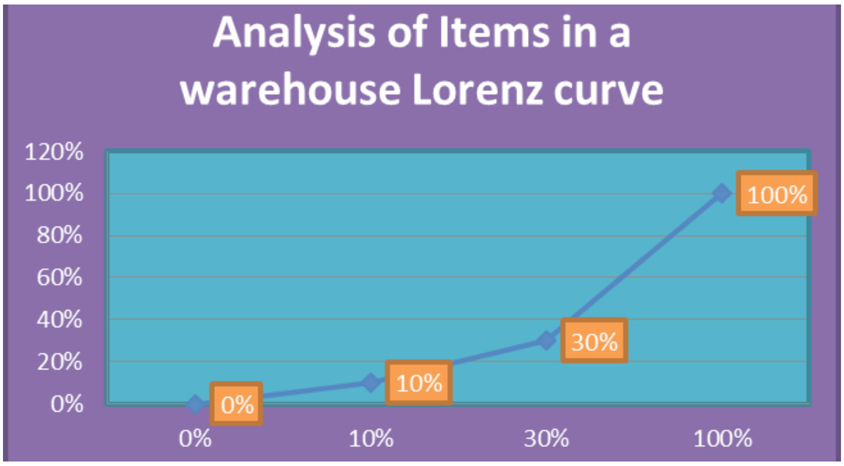
Example 8.23

The table below shows date items recorded in a warehouse railway good shed,

Cumulative

Classification	**% of Total Value**	**% of Total Units**
	0%	0%
A	70%	10%
B	90%	30%
C	100%	100%

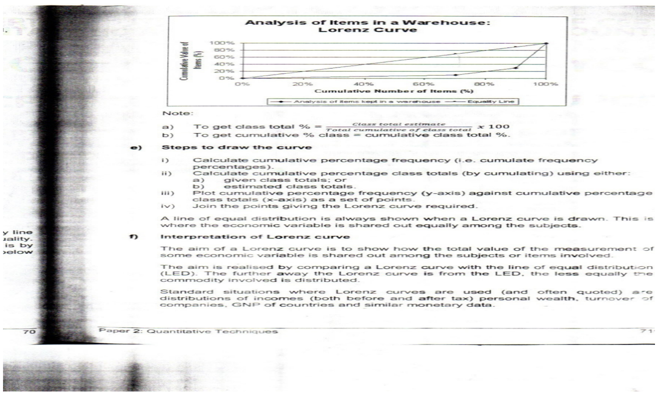
Draw a Lorenz curve to illustrate this date.



8.12.3. The Equality Curve

To better illustrate the situation, it is common to add what is called an equality line to a Lorenz curve (LC): this is the line that shows a 1:1 relationship….total equality. Consequently, the equality line shows us how equal or unequal a situation is by illustrating the distance between the pareto curve and the equality line . Figure below illustrates the equality line:

Analysis of items in a warehouse Lorenz Curve



Note:

a) To get class total % = $\frac{\textit{Class total estimate}}{\textit{Total cumulative of class total}} x\ 100$

b) To get cumulative % class = cumulative class total %.

e) **Steps to draw the curve**

i) Calculate cumulative percentage frequency (i.e. cumulate frequency percentages).

ii) Calculate cumulative percentage class totals (by cumulating) using either:

a) given class totals; or

b) estimated class totals.

iii) Plot cumulative percentage frequency (y-axis) against cumulative percentage class totals (x-axis) as a set of points.

iv) Join the points giving the Lorenz curve required.

A line of equal distribution is always shown when a Lorenz curve is drawn. This is where the economic variable is shared out equally among the subjects.

f) **Interpretation of Lorenz curve**

The aim of a Lorenz curve is to show how the total value of the measurement of some economic variable is shared out among the subjects or items involved.

The aim is realised by comparing a Lorenz curve with the line of equal distribution (LED). The further away the Lorenz curve is from the LED, the less equally the commodity involved is distributed.

Standard situations where Lorenz curves are used (and often quoted) are distributions of incomes (both before and after tax) personal wealth, turnover of companies, GNP of countries and similar monetary data.

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Note:

a) To get class % = $\frac{\textit{class total estimate}}{\textit{total cumulative of class}}\ x\ \frac{\textit{class total estimate}}{\textit{total cumulative of class}}\ x\ 100$

b) To get cumulative % class= cumulative class total %.

Steps to draw the curve

i) Calculate cumulative percentage frequency (i.e. cumulate frequency percentages).

ii) Calculate cumulative percentage class totals (by cumulating) using either:

a) Given class totals; or

b) Estimated class totals.

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A line of equal distribution is always shown when a Lorenz curve is drawn. This is where the economic variable is shared out equally among the subject

Interpretation of Lorenz curve

The aim of a Lorenz curve is to show how the total value of the measurement of some economic variable is shared out among the subjects or items involved.

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Standard situations where Lorenz curves are used (and often quoted) a'e distributions of incomes (both before and after tax) personal wealth, turnover of companies, GNP of countries and similar monetary data

Uses of Lorenz curves

They are used to show inequalities in connection with matters, such as:

- incomes in the population;
- tax payment of individuals in the population;
- industrial efficiencies;
- industrial outputs;
- examination marks; and
- customers and sale

Example 8. 24

The distribution below gives the personal wealth of a certain cross-section of the population of Rwanda for a particular year.

Personal wealth (Frw)	Number of persons (000,000)	Total personal wealth (Frw 000m)
	(f)	(V)
0-2000	19	2.4
2000 - 5000	26	7.8
5000- 10000	74	55.5
10000- 15000	41	49.2
15000-20000	16	25.7
20000-25000	8	16.8
25000-50000	5	15.0
50000 and over	1	6.3
Total	**190**	**178.7**

Required:

Construct:

i) a table for drawing a Lorenz curve; and

ii) a Lorenz curve.

Solution:

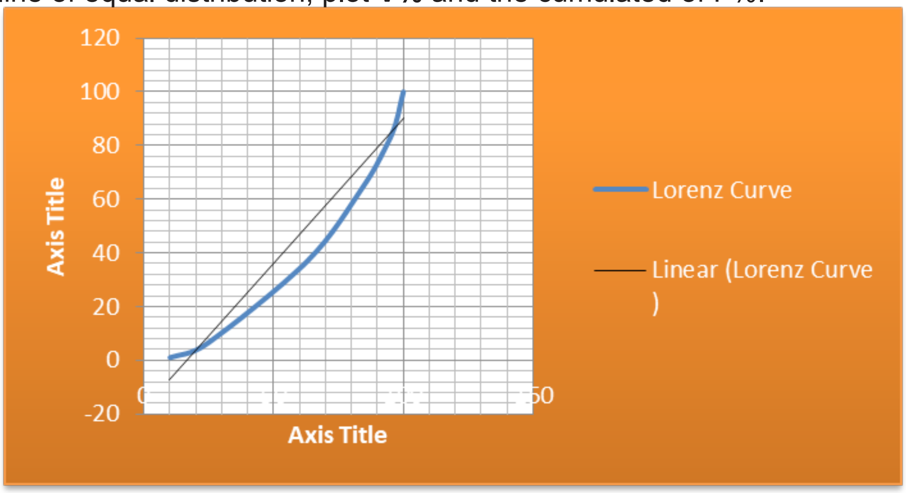
The class totals, in this case total personal wealth, are given. Thus, they do not need to be estimated, the following abbreviations have been used in the calculations that are shown in the table below; f= frequency = number of persons; F = cumulative f; v = values of class totals = estimate of total personal wealth; V =

cumulative v.

Table for drawing a Lorenz curve

f	F	F%	V	V	V%
19	19	10	2.4	2.4	1
26	45	24	7.8	10.2	6
74	119	63	55.5	65.7	37
41	160	84	49.2	114.9	64
16	176	93	25.7	140.6	79
8	184	97	16.8	157.4	88
5	189	99	15.0	172.4	96
1	190	100	6.3	178.7	100

Note: When you plot F% on the vertical axis and **V%** on the horizontal axis you get the curve. To get the line of equal distribution, plot **V%** and the cumulated of F%.



8.13. Limitations

There is no single recommended 'correct' graph to use for a given set of data. Some graphs are usually more appropriate than others in terms of bringing out various attributes of the data in relative terms while others are good for actual comparisons. Misrepresentation of data, either by accident or design, is caused by not following accepted standards. Common misrepresentations often result from not showing breaks of scale, faulty or missing scaling on a numeric axis and confusion over linear, area and volume pictograms.

8.13.1. Limitations for each type of graph

i) Pictograph:

- Hard to quantify partial
- icons. Icons must be of consistent size.
- Best for only 2- 6 categories
- Very simplistic.

ii) Line plots:

- Not as visually appealing
- Best for under 50 data values.
- Needs small range of data.

iii) Pie charts:

- No exact numerical data
- Hard to compare two data sets.
- Other categories can be a problem.
- Total unknown unless specified.
- Best for three to seven categories.
- Use only with discrete data.

iv) Histogram:

- Cannot read exact values because data is grouped into categories.
- More difficult to compare two data sets.
- Use only with continuous data.

v) Bar graph:

- Graph categories can be reordered to emphasise certain effects.
- Use only with discrete data.

vi) Line graph:

- Use only with continuous data.

vii) Frequency polygon:

- Anchors at both ends may imply zero as data points
- Use only with continuous data.

Viii) Lorenz curve:

- When the Lorenz curves of two countries intersect, it would be difficult to judge the difference of income distribution in that case.

ix) Z-charts:

- Negative scores have bad connotation.
- Two decimal places cumber same.
- Not everyone understands them that easily.

8.14. Self-test questions

Question 8.1

A census of workers employed in a number of enterprises established in a certain suburb was taken. The results were:

30	45	40	30
35	40	35	30
45	30	35	40
35	30	30	35
45	45	40	30
35	45	30	35

Required

a) Prepare a frequency table for the data.

b) State the most common enterprise in the suburb

Solution

a) Frequency table

Value	Frequency
30	8
35	6
40	4
45	6

a) Most common enterprise has 30 workers.

Question 8.2

A company took head count of all its workers. There were 48 workers in total and the following list shows their ages:

34	28	22	36	27	18	52	39	42	29	35	31
27	22	37	34	19	20	57	49	50	37	46	25
18	37	42	53	41	51	35	24	33	41	53	60
18	44	38	41	48	27	39	19	30	61	54	48

Required:

a) Prepare a table grouping the data into classes from 10-19, 20 -29, 30 -39 etc.

b) State the group with most workers.

Solution:

a)

Class intervals	Frequency (f)
10-19	5
20-29	10
30-39	14
40-49	10
50-59	7
60-69	2
	48

b) The age group with the most workers is 30 to 39.

Question 8. 3

The data shows the tax returns in million Rwandan Francs to the Rwanda Revenue Authority by a number of

Companies in a certain year.

34	12	45	23	12	18	26	41	48	23	47	11
7	15	31	28	54	32	63	8	32	21	29	45
15	9	20	37	43	27	30	17	14	26	34	24
18	16	35	32	27	14	30	22	31	40	17	24

Required:

a) Prepare a table of grouped data starting with class 1-10.

b) State the group of companies with the highest tax returns to the Revenue Authority.

Solution

a)

Class intervals

1 - 10

11-20

21 -30

31 -40

41 -50

51-60

61 -70

Class mark (x)

5.5

15.5

25.5

35.5

45.5

55.5

65.5

Frequency (f)

3

13

14

9

7

1

1

$\sum f=48$

b)

Group with highest number of tax returns is 21 to 30.

Required:
Question 8.4

The table below shows installed electricity capacity (MW) in Rwanda.

Plant name	2016	2017	2018
Hydro electricity	315	328	352.5
Thermal electricity	200	150	170
Biogaz electricity	12	17	17

Find the total:

a) production of each type of electricity in the period given.

b) electricity output for each yea

a) The total production of each type of electricity in the period given is shown in the table above, namely:
   - Hydroelectricity = 995.5;

- Thermal electricity = 520; and
- Biogas electricity = 46.

b) The electricity output for each year is also shown in the table above, namely:
- 2008-527;
-2009- 495; and
- 2010-539.5.

Solution:

Plant name	2016	2017	2018	Total
Hydro electricity	315	328	352.5	**995.5**
Thermal electricity	200	150	170	**520**
Biogas electricity	12	17	17	**46**
Total	**527**	**495**	**539.5**	

Question 8.5

A survey was made on the ages of vehicles which took the MOT safety test.

	Failed test	Passed test	Total
"Less than 4 years old	15	96	-
4 to 7 years old	43	67	—
Over 7 years	97	88	—
Total	-	-	-

Required:

a) Copy and complete table: and
a) Find the:
   i) Total number of vehicles in the survey.
   ii) Number of vehicles that failed the test.
   iii) Percentage of vehicles over 7 years old that passed the test.
   iv) Percentage of vehicles less than 4 years old that failed the test.
   v) The percentage of vehicles aged 4 to 7 years.

Solution:

a) The completed table is shown below:

	Failed test	Passed test	Total
Less than 4 years old	15	96	111
4 to 7 years old	43	67	110
Over 7 years	97	88	185
Total	= 155	=251	=406

b) i) Total number of vehicles in the survey is 406.
   ii) Number of vehicles that failed the test is 155.
   iii) Percentage of vehicles over 7 years old that passed the test is 47.6%.
   iv) Percentage of vehicles less than 4 years old that failed the test is 13.5%.
   v) The percentage of vehicles aged 4 to 7 years is 27.1%.

Question 8.6

Lakalacor Municipality has a total of 200 employees in four departments as follows: Social Services 100, Administration 30, Revenue Collection 50 and 20 in Law Enforcement. Furthermore, it is stated that 60 of the employees in Social Services are female, five female staff are in Law Enforcement, while Revenue

Collection and Administration have 20 and 10 male employees, respectively.

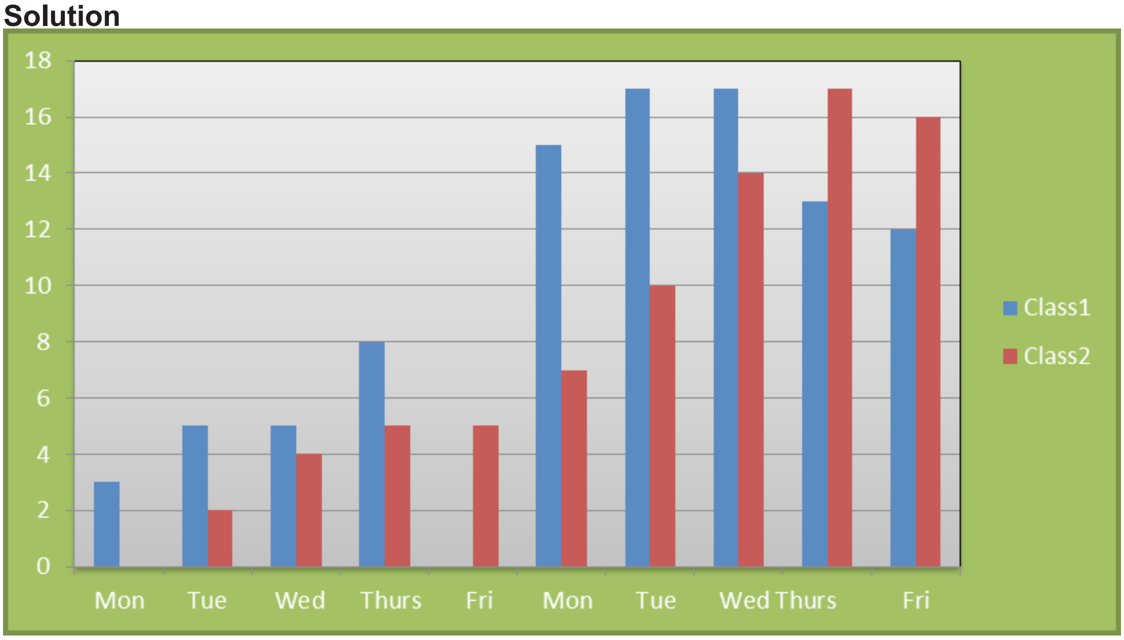
Required:
Construct a table representing the above data.

	Social services	**Administration**	**Revenue Collection**	**Law Enforcement**
Female	60	20	20	5
Male	40	10	30	15
Totals	100	30	50	20

Question 8.7
Draw a compound bar chart to illustrate the following: the data represents the absences in two classes at school during Bahoneza campaign April 2019.

Day	Mon	Tue	Wed	Thurs	Fri	Mon	Tue	Wed	Thurs	Fri
Class1	3	5	5	8	1 0	15	17	17	13	12
Class2	0	2	4	5	5	7	10	14	17	16

Solution



Question 8.8
The following pie chart shows a survey of the numbers of cars, buses and motorcycles that passes a particular junction. There were 150 buses in the survey.

Motorcycles

210°

Buses 30°

Cars 120°

a. What fraction of the vehicles were motorcycles?
b. What percentage of vehicles passing by the junction were cars?
c. Calculate the total number of vehicles in the survey?

d. How many cars were in the survey?

Solution:

a) Fraction of motorcycles

$\frac{\text{angle of sector}}{360} = \frac{210^0}{360^0} = \frac{7}{12}$

b) To convert the angle of a sector into a percentage, we use the formula:

$= \frac{\text{angle of sector}}{360} x100\%$

Percentage of care

$\frac{120^0}{360^0} x100\% = 33\frac{1}{3}\%$

c) Let x be the total number of vehicles

$\frac{30^0}{360^0} Xx \frac{30^0}{360^0} Xx=150$

X=150*360°

30°

X=1,800

The total number of vehicles is 1,800

d) Number of cars

$\frac{120^0}{360^0} x1800 = 600$

Question 8.9

The following line graph shows the total number of animals in a zoo.

a) In which year did the zoo have the largest number of animals?
b) What is the percentage increase of animals in the zoo from 1999 to 2001?

Solution:

a) The zoo had the largest number of animals in 2002
b) The percentage increase of animals in the zoo from 1999 to 2001 is

c) $\frac{500-200}{200} X100\% = 150\% \frac{500-200}{200} X100\% = 150\%$

Question 8.10

The production of each manufacturing department of your company is monitored weekly to establish productivity bonuses paid to the members of that department. 250 items have to be produced each week before a bonus will be paid. The production in one department over a forty-week period is shown below:

382 367 364 365 371 370 372 364 355 347 354 359 359 360 357 362 364 365 371 365 361 380 382 394 396 398 402 406 437 456 469 466 459 454 460 457 452 451 445 446

Required:

a) Form a frequency distribution of five groups for the number of items produced per week.
b)Construct the Ogive or Cumulative frequency diagram for the frequency distribution established

Question 8.11

The production for each manufacturing department of your company is monitored weekly to establish productivity bonuses paid to the members of that department.250 items have to be produced each week before a bonus will be paid. The production in one department over a forty-week period is shown below:
382 367 364 365 371 370 372 364 355 347 354 359 359 360 357 362 364 365 371 365 361 380 382 394 396 398 402 406 437 456 469 466 459 454 460 457 452 451 445 446

Required:

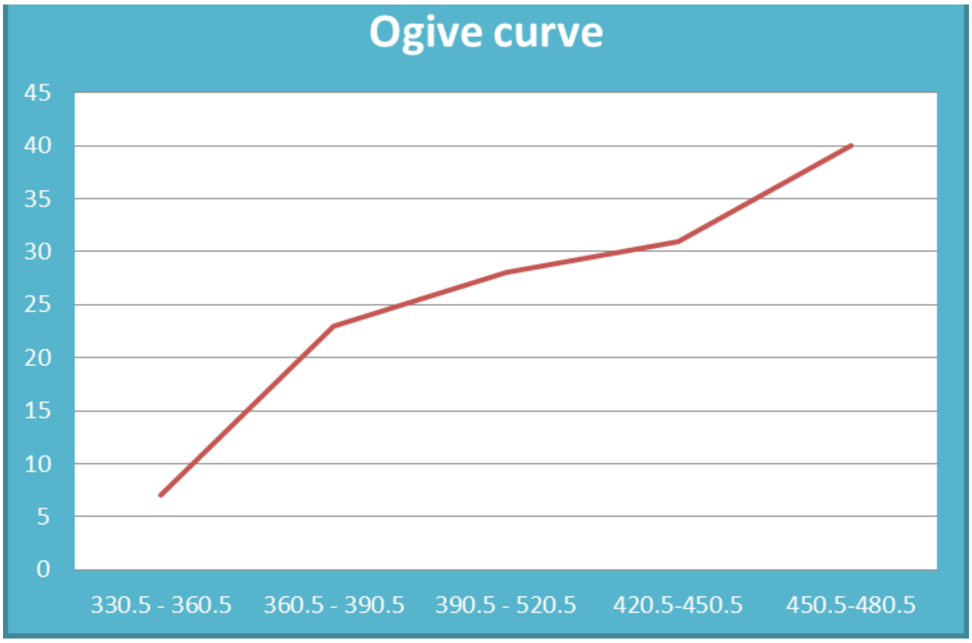
a) Form a frequency distribution of the five groups for the number of items produced per week.

b) Construct the Ogive or cumulative frequency distribution diagram for the frequency distribution established a)

Solution:

a) The table below shows the frequency distribution for the five groups of the number of items produced per week.

Class interval	Frequency	Class limits	Cumulative frequency
331 – 360	7	330.5 - 360.5	7
361 – 390	16	360.5 - 390.5	23
391 -420	5	390.5 - 520.5	28
421 -450	3	420.5-450.5	31
451 – 480	9	450.5-480.5	40
	40		



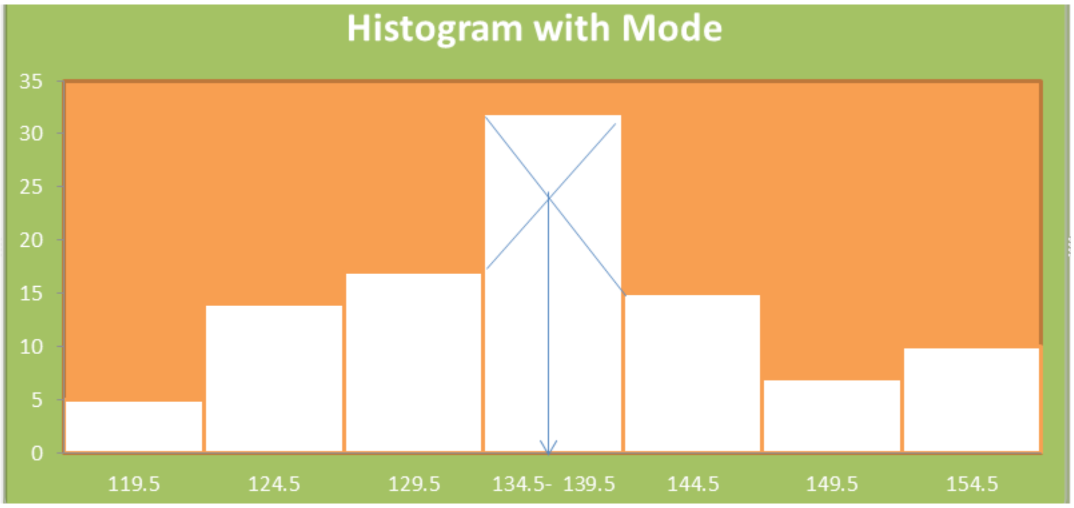
Question 8.12

An athletics trial was conducted in the highland training camp in Kenya. The performance of the athletes who took part on a certain day is summarized in the following table, capturing the time in minutes that the athletes registered:

Time (miniute)	(not of athletes)
120- 124	5
125- 129	14
130- 134	17
135- 139	32
140- 144	15
145- 149	7
150- 154	10

Required;

Construct a histogram to represent the above data and use it to estimate the modal time, in minutes, registered by the athletes



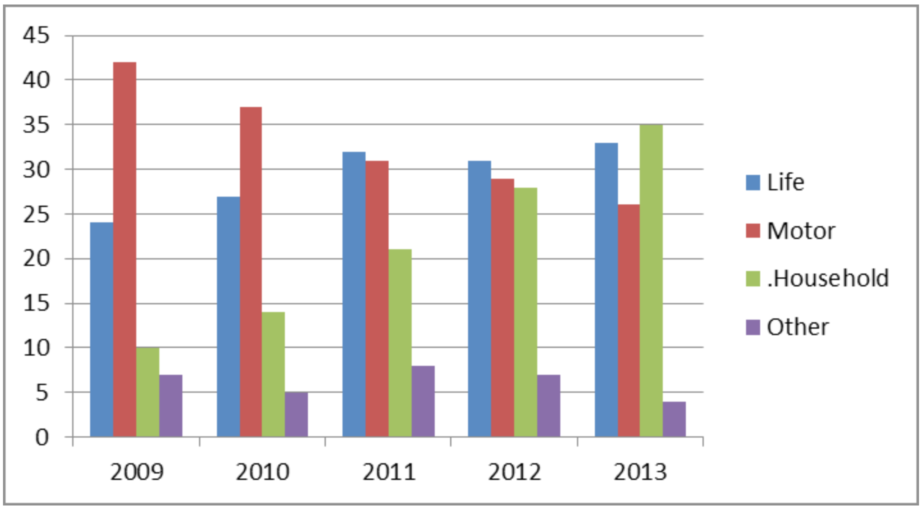
Modal time from graph = 137 minutes **Question 8.13**

The following table shows the number of insurance policies by class of business (numbers expressed in thousands), issued by an insurance company during the years 2009 - 2013.
Numbers in years

Policy type	2009	2010	2011	2012	2013
Life	24	27	32	31	33
Motor	42	37	31	29	26
.Household	10	14	21	28	35
Other	7	5	8	7	4

Required:

a) Carefully draw a suitable chart to illustrate the data
b) What are the advantages and disadvantages of your form of representation



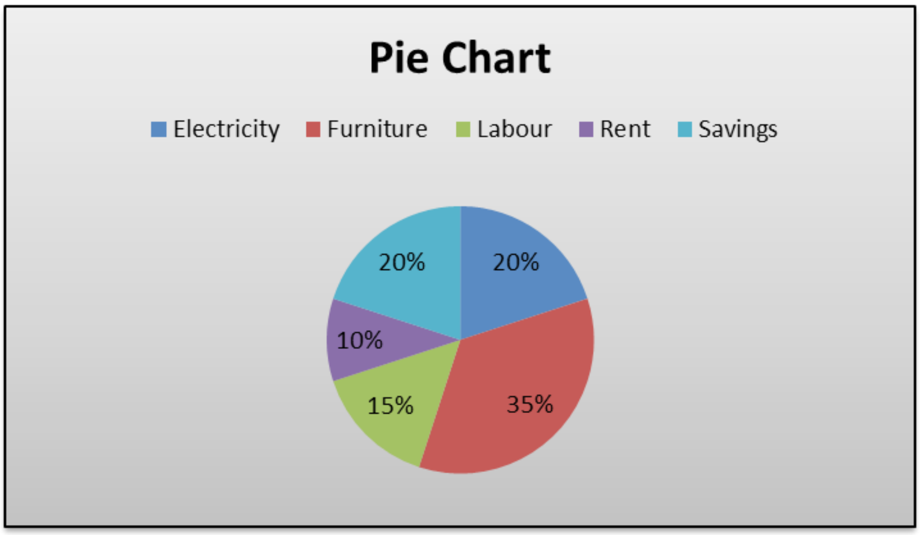
Advantages:

- Visual display
- Can accommodate as many sub components into one bar
  Disadvantages: Take time to work out individual components and their member data totals

Question 8.14

Mrs. A Joy made a profit of Frw 5,000.000 from her company. Her expenditure was as follows

Electricity	20%
Furniture	35%
Labour	15%
Rent	10%
Savings	20%



Question 8.15
Using the data of the table below, estimate the total value of the repayments made by the members of each of the nine classes of the distribution.

a) Use these same data to draw up a table of cumulative percentage number of shop Owners against cumulative percentage repayments.
b) Use this table to construct a Lorenz curve.

Repayment (‘000)	Number of shop owners
x< 5	7
5 < x < 9	60
9 < x < 13	101
13 < x < 17	105
17 < x < 21	70
21 < x <25	30
25 < x < 29	10
29 < x < 33	8
33 < x < 37	9

CHAP 9.

MEASURES OF LOCATION (CENTRAL TENDENCY)

9.1. Study objectives

By the end of this chapter, you should be able to.

- calculate measures of central tendency such as the mean, median, mode using different techniques / formulae;
- calculate measures of location weighted mean, geometric mean and harmonic mean;
- determine median and mode by graphical method; and
- give advantages and disadvantages of measures of location.

9.2. Measure of location (central tendency)

Measures of central tendency are methods (or values) used to describe the middle or a central characteristic of a set of data that best represents all the numbers in the sample or population. They show the average position of a given distribution of data. They are values which lie between the two extreme observations in the distribution that are used to summarize data by trying to find one number. In general, they are referred to as averages. In statistics, an average is defined as the number that measures the central tendency of a given set of numbers. Some terms may be used to describe the center of a set of data. The most common value in a set of data is the mode, a typical middle value is a median and the average is the arithmetic mean. These measures provide important information which allows calculations that give a relationship of the middle value in a distribution. There are other measures of average which include weighted mean, harmonic mean and geometric means.

Measures of central tendency are very useful are very useful:

- For describing a distribution in a concise manner.
- For a comparative study of different distributions.
- For computing various other statistical measures, such as dispersion, skewness, kurtosis, etc.

9.3. Properties of measures of central tendency

A good measure of central tendency should have the following properties:

- Should be clearly defined.
- Should be based on all observations.
- Easy to calculate
- Should not be influenced by sampling fluctuations.
- Should be able to be used for further algebraic treatment.
- Should not be affected by extreme value. The most common measures of location include: Arithmetic mean (mean), median, mode, weighted mean, geometric mean and harmonic mean

9.4. Measures of central tendency

9.4.1. Arithmetic mean (mean)

The arithmetic mean (mean) of a distribution of data is the sum of all the values divided by the number of values in that distribution. Mean is a method of representing the whole data in a distribution by one value. It is the most widely used measure of central tendency. If you use a wrong average, you will get inaccurate results. The choice of average depends upon the distribution of the data and the purpose for which it is to serve.

Arithmetic mean is a good measure, but cannot be effectively used when:

- The distribution has open ends.
- The distribution is highly skewed.
- Average is taken for ratios or percentages.

i) Properties of the mean:

- It gives a general picture of the whole distribution.
- It represents the entire distribution and hence summarizes the whole data in the distribution.
- It can be used for decision making.

ii) Computation methods of mean

To find the mean of a set of individual observations, add their values and divide by number of observations. If n are observations $x_1, x_2, x_3, \ldots, x_n$ then the mean is

$\bar{x} = \frac{x_1+x_2+x_3+\ldots+x_n}{n}\bar{x} = \frac{x_1+x_2+x_3+\ldots+x_n}{n}$ Mean is often by the formula $\bar{x} = \frac{\sum x}{n}\bar{x} = \frac{\sum x}{n}$ where $\sum x \sum x$ means sum or total of x and n is the number of x values.

Example 9.1

A student sat for six course work tests in the first semester at university of Rwanda and scored the following marks out of 50: 23, **17,** 30, 27, 29, and 36.

Find the mean mark.

Solution:

Total marks = 23 + 17 + 30 + 27 + 29 + 36 = 162

$$\bar{x} = \frac{23 + 17 + 30 + 27 + 29 + 36}{6} = 27$$

c) **Mean of a discrete frequency distribution**

The values are first arranged in the frequency distribution table and mean is Calculated using the formula mean $\bar{x} = \frac{\sum fx}{\sum f}\bar{x} = \frac{\sum fx}{\sum f}$

Example 9.2

Find the arithmetic mean of the numbers:

10, 11, 12, 12, 13, 11, 12. 11, 12. 12, 11, 12, 11, 13, 10, 11, 12, 12, 14, 11

Solution:

Value (x)	Frequency (f)	fx
10	2	20
11	7	77
12	8	96
13	2	26
14	1	14
	$\sum f \sum f = 20$	$\sum f \sum f$**x** = 243

Solution:

The mean $\bar{x} = \frac{\sum fx}{\sum f}\bar{x} = \frac{\sum fx}{\sum f}$ **= 243/20** =12.06

Calculation of the mean when data is presented in a frequency distribution:

Example 9.3

The numbers of children who go to school per family on a housing estate were recorded as follows.

No. of children	0	1	2	3	4
No. of families	12	15	5	2	1

Required:

Find the mean number of children per family who go to school.

No. of children (x)	No. of families (f)	(Wx)
0	12	0x12=0
1	15	1x15=15
2	5	2 x5=10
3	2	3 x2 =6
4	1	4 xl =4
	$\sum f \sum f$= 35	$\sum fx \sum fx$=35

Solution:

$$\overline{x} = \frac{\sum fx}{\sum f} = \frac{35}{35} = 1$$

This means on average 1 child goes to school per family**.**

e) Calculation of the mean using the working mean (assumed mean)

In this method, the deviation (d) of each variable from the assumed mean should be found. Any value among or outside the distribution can be taken as the assumed mean. Deviations are then multiplied by the corresponding frequency (f). Then the total of (fd) is divided by total frequency, the result is added or subtracted from the assumed mean to give the required mean.

$$\overline{x} = A + \frac{\sum fd}{\sum f}$$

The formula for the assumed mean method, where A = assumed mean, total frequency is$\sum f \sum f$and $d d$is deviations from the assumed mean $(d = x - A).(d = x - A)$.

Example 9.4

The following table shows the average saving contribution per worker per months in 1000s of Rwandan Francs in a saving scheme.

Savings in 1000s Frw	500	600	700	800	900	1000	1100
No of persons	1	3	5	7	6	2	1

Required:

Calculate the arithmetic mean saving for the workers using 800 as assumed mean. Calculate the men

Savings in 1000s Frw	d=(x-A)	No of persons	fd
500	-300	1	-300
600	-200	3	-600
700	-100	5	-500
800	0	7	0
900	100	6	600

1000	200	2	400
1100	300	1	300
		25	-100

Using the working mean method $\overline{x} = A + \frac{\sum fd}{\sum f}\overline{x} = A + \frac{\sum fd}{\sum f}$ **the required mean =Assumed mean +Mean deviation**

$$\overline{x} = 800 + \frac{-100}{25} \Rightarrow \overline{x} = 796$$

f) Calculating the mean of a grouped distribution

Sometimes a distribution is so varied that it is cumbersome to work with the original raw data. Instead, the data is grouped into classes.

g) How to deal with values when they are in classes

Since there is a spread in each group, it is impossible to determine the mean. We can only find an approximation to the mean. It is assumed that each value within a class interval is equal to the midpoint of that group and also that all class intervals are of the same size. There are usually two types of class intervals continuous and discrete intervals. A continuous interval leaves no gap between the class intervals such as 5-10, 10-15, 15 -20 etc. and a discrete interval takes the form 4-9. 10 - 14, 15-19, etc.

h) The arithmetic mean of a grouped discrete frequency distribution

Example 9.5
Given the frequency distribution below

Value	Frequency
71 -75	1
76-80	2
81 -85	4
86-90	4
91 -95	5
Total frequency	**16**

Required:

Find the arithmetic mean of grouped frequency distribution

Solution:

Method 1: Find midpoints or mid-value of each class. Then calculate the mean using the mid values, here after referred to as class marks

Value	frequency	class Marks	fx
71 -75	1	73	73
76-80	2	78	156

81 -85	4	83	332
86-90	4	88	352
91 -95	5	93	465
Total frequency	**16**		1378

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{1378}{16} = 86.125$$

Method 2

By step coding using an assumed mean and a scale

First we obtain class mark or midpoints or mid-value of each class then determine the assumed value from class mark in this example let the assumed mean be 83, then calculate the deviations from the assumed mean in the as bellow

Value	frequency (f)	Class Marks (x)	D=X-83	fD	U=(x-83)/5	FU
71 -75	1	73	-10	-10	-2	-2
76-80	2	78	-5	-10	-1	-2
81 -85	4	83	0	0	0	0
86-90	4	88	5	20	1	4
91 -95	5	93	10	50	2	10
Total frequency	**16**		∑f d=50		∑f u=10	

The mean $\bar{x} = A + \frac{\sum fu}{\sum f} * h = 83 + \frac{10}{16} * 5 = 86.125\ or\ simply\ \bar{x} = A + \frac{\sum fd}{\sum f} = 83 + \frac{50}{16} = 86.125$

$$\bar{x} = A + \frac{\sum fu}{\sum f} * h = 83 + \frac{10}{16} * 5 = 86.125\ or\ simply\ \bar{x} = A + \frac{\sum fd}{\sum f} = 83 + \frac{50}{16} = 86.125$$

i) Calculating the arithmetic mean of grouped continuous frequency distribution

The method of calculating the mean grouped continuous frequency distribution is the similar to that grouped discrete frequency distribution

Example 9.6

The heights of 30 flowers in in the garden were recorded in cm and grouped as shown in the table bellow

Height(cm)	5-15	15-25	25-35	35-45	45-55
Frequency(f)	6	4	15	3	2

Required: Find the mean of height of the flower

Height(cm)	Frequency(f)	Class Marks (x)	fx
5-15	6	10	60
15-25	4	20	80
25-35	15	30	450
35-45	3	40	120
45-55	2	50	100
	∑f =30	∑f x=810	

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{810}{30} = 27$$

Sometimes the classes of frequency distribution may be in the form less or equal form as shown in the example. In such a case the class includes all values except the upper limit of the individual class.

Example 9.7

Given the continuous distribution table for mass (gm) below:

Mass (gm)	Frequency
100 ≤≤x < 140	16
140 ≤≤ x < 160	46
160 ≤≤x < 180	76
180 ≤≤x < 200	54
200 ≤≤ x < 220	40
220 ≤≤ x <240	10
240 ≤≤ x <300	18

Mass (gm)	Frequency	Class mark (x)	D=x-A	u=(X-A)/h	fu
120<x<140	16	120	-70	-3.5	-56
140<x<160	46	150	-40	-2	-92
160<x<180	76	170	-20	-1	-76
180<x<200	54	190	0	0	0
200<x<220	40	210	20	1	40
220<x<240	10	230	40	2	20
240<x<260	18	250	60	3	54
					-110

$$\bar{x} = A + \frac{\sum fu}{\sum f} * h = 190 + \frac{-110}{260} * 20 = 181.5$$

j) Calculation of arithmetic mean in case of open ended classes

For calculating the mean, we have to find out the mid points (class marks). Therefore, to calculate the mean we have to make assumptions about unknown class limits. For the first class width is taken to be equal to the length of the immediately succeeding class. Similarly, the last class is taken to be equal to the length of immediate preceding class.

Examples 9.8. The table below shows the number of workers of Agriculture Cooperatives in Kayonza District

Number of Workers	Below 50	50-100	100-150	150-200	200-250	250-300	above 300
No. of Cooperatives	13	9	0	7	4	5	2

Required: Calculate the mean of workers in Kayonza District.

Since the lower and upper class boundaries are open it is difficult to obtain the mean. But we assume all classes have equal class intervals hence we calculate the mean in the usual way.

Number of Workers	No. of Cooperatives(f)	Class mark	fx
0- 50	13	25	325
50-100	9	75	675
100-150	0	125	0
150-200	7	175	1225
200-250	4	225	900
250-300	5	275	1375
300-350	2	325	650
	40		5150

$$\bar{x} = \frac{\sum fx}{\sum f} \Longrightarrow \bar{x} = \frac{5150}{40} = \mathbf{128.75} \approx \mathbf{129}\ \boldsymbol{workers}$$

Example 9.9

The data below shows the income and number of employees at GIRINKA Company in Muhanga District

income in Frw '000'	employees
Below 30	15
Below 40	36
Below 50	61
Below 60	76
Below 70	87
Below 80	95
Above 80	5

Example 9.10. Given that the total income of highest group is Frw 435000 and none earns below Frw 20000

Required: Calculate the mean income

Solution:

Since it is given that none earns below Frw 20000 the first class interval is assumed to be 20-30. And be assumed that all class interval are equal. Hence the cumulative frequency distribution has to be changed into frequency distribution as shown below

income in Frw '000'	employees(f)	Class marks(x)	fx
20-30	15	25	375
30-40	21	35	735
40-50	25	45	1125
50-60	15	55	825
60-70	11	65	715
70-80	8	75	600
80-90	5	85	425

	$\sum f = \sum f = 100$		$\sum fx \sum fx$ =4800

$$\bar{x} = \frac{\sum fx}{\sum f} \Longrightarrow \bar{x} = \frac{4800}{100} = 48$$

Example 9.11

The frequency table below shows heights of filling cabinets.

Height cm	140-150	150-160	160-170	170-180	180-190	190-200	Total
Frequency	5	F1	20	F2	6	2	52

Given that the mean of the frequency distribution is $166\frac{9\ \ 9}{5252}$

Required:

Find the values of $F_1$and F_2.

9.5. Median

Median is the middle value when all values in a distribution are arranged in order of size. If the number of values is odd the median becomes the exact value of the middle term, but when the number is even the median becomes the average of the two middle terms.

Median of a discrete distribution: odd number of individual items or values.

Example 9.12

The weights of seven police recruits are 80, 86,91, 101, 109, 120 and 106kgs. Find the median. Arrange the data in order: 80, 86, 91, 101, 106, 109, 120 Select t the middle value 101 is the median. An even number of individual items or values.

Example 9.13

The numbers of pages in 10 textbooks were recorded as follows: 414, 398, 402, 399, 40C 405, 395, 401, 412, and 407. Find the median.

Arrange the values in order of size: 395. 395, 398, 399, 400, 402, 405, 414

Median = $\frac{399+400}{2} = 399.5\frac{399+400}{2} = 399.5$

a) **Median of frequency distribution**.

The median is calculated using a cumulative frequency distribution. The method is illustrated as follows:

Example 9.14

Score/value	Frequency (f)	Cumulative frequency
1	1	1
2	2	3
3	4	7
4	5	12
5	6	18
6	8	26
7	11	37

8	8	45
9	3	48
10	2	50

Write in columns the values of the variable and their frequencies

Calculate a cumulative frequency column

Locate the middle data for n items, this is the item $\frac{(n+1)}{2}\frac{(n+1)}{2}$

The median is the value of the middle data item. The middle student in a group of 50 students is a student number $\frac{(50+1)}{2}\frac{(50+1)}{2}$ =25.5 There is no such student so this. Can be interpreted half way between 25th and 26th students and both scores are the same. The median in the above data scores = 6

b) Median of a continuous (grouped) frequency distribution

Sometimes you may need to estimate the median from a frequency distribution. This is done by first determining which interval contains the median and then determining the proportion of the interval needed to get up to halfway point. This sometimes is called interpolated median or estimated median. The method for calculating the interpolated median is given by the formula:

$$Median = L_{me} + \left(\frac{\frac{n}{2} - F}{f}\right)C$$

Where $L_{me}L_{me}$ is the lower class boundary of the interval containing the median

C is the class size (Width) of the median class

n is number of frequency distribution of total frequency in the distribution

f is the frequency of the median class

F is the cumulative frequency below median class interval

Example 9.15

The table below shows the price variable of the real estate in 1000s of Frw data

Price	150-200	200-250	250-300	300-350	350-400	400-450	450-500	500-550
Frequency	2	13	23	27	26	19	12	2

Required: estimate the median using interpolation formula

Solution

Using this formula

$$Median = L_{me} + \left(\frac{\frac{n}{2} - F}{f}\right)C$$

	Price	Frequency	Cumulative Frequency
	150-200	2	2
	200-250	13	15
	250-300	23	38
Median Class	300-350	27	65
	350-400	26	91
	400-450	19	110

450-500	12	122
500-550	2	124

The median class is obtained by taking $X_{n/2}X_{n/2}$ so is $X_{124/2} = X_{62}X_{124/2} = X_{62}$the class lies in the cumulative frequency of 65 hence class is 300-350

$$Median = 300 + \left(\frac{\frac{124}{2} - 38}{27}\right)50 = 344.4$$

Example 9.16

Two hundred steel bars produced by Steel RW were measured and results are summarized in the following table:

Length(m)	20-22	23-25	26-28	29-31	32-34	35-37
Frequency(f)	8	34	82	50	14	12

Required: by Calculation Find the median

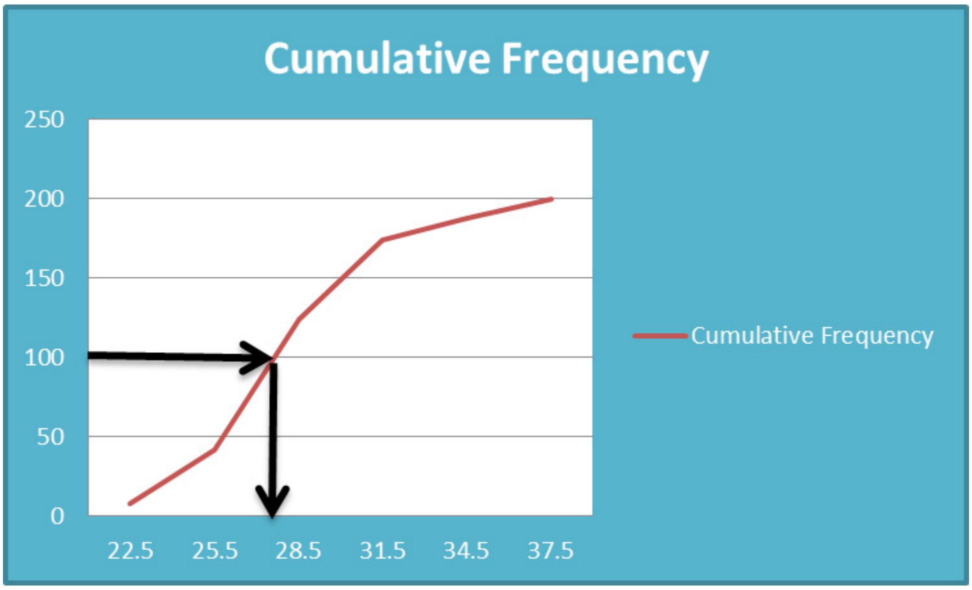
Construct cumulative frequency curve and use it to estimate Median

Using class limits since it is discrete distribution

	Length(m)	Class limits	Frequency(f)	Cumulative Frequency
	20-22	19.5-22.5	8	8
	23-25	22.5-25.5	34	42
Median Class	26-28	25.5-28.5	82	124
	29-31	28.5-31.5	50	174
	32-34	31.5-34.5	14	188
	35-37	34.5-37.5	12	200

$$Median = 25.5 + \left(\frac{\frac{200}{2} - 42}{82}\right)3 = 27.6$$

By constructing the cumulative frequency curve



The median is 27.6 m

Remarks: we cannot find the exact median for a grouped frequency distribution since we do not have individual values to arrange in either ascending or descending order. However an approximate value for median can be obtained from the interpolation formula or a cumulative curve

9.6. Mode

For ungrouped data, the mode is easiest to find, since it merely involves locating the values with the highest frequency in the distribution. For a grouped disribution, it is not possible to stae the mode accurately but it is easy to to find a modal class, that is the class with the highest frequency. However it is possible to obtain an estimate of the mode of a grouped distribution in two ways:

i) By construction using Histogram
ii) By calculation using interpolation formula.

interpolation formula is

$$Mode = L_{mo} + \left(\frac{D_1}{D_1 + D_2}\right)C$$

Where L_{mo} isL_{mo} is lower boundary of median class

$D_1 D_1$ is the difference between frequency of modal class and frequency of Previous class

$D_2 D_2$is the difference between frequency of modal class and frequency of next class

C is the modal class interval (Width)

Example 9.17
The table shows the heights of students in one age group

Height of students (cm)	**Frequency**
152 ≤ x < 156	4
156 ≤ x < 160	8
160 ≤ x < 164	8
164≤ x < 168	14

168 ≤ x < 172	10
172 ≤ x < 176	12
176 ≤ x < 180	4

Required:
Estimate the mode by interpolation formula.
Solution
$L_{mo}L_{mo}$=162 C=4
D_1D_1=14-8=6 D_2D_2=14-10=4
The modal class is 162 ≤ x < 164which frequency is 14

$$Mode = 164 + \left(\frac{6}{6+4}\right)4 = 166.4$$

Example 9.18
In case of estimating the mode graphically we use a histogram as follows: Construct a histogram of the frequency distribution,
Draw lines i) from the top left corner of the modal class to the top corner of the next class.

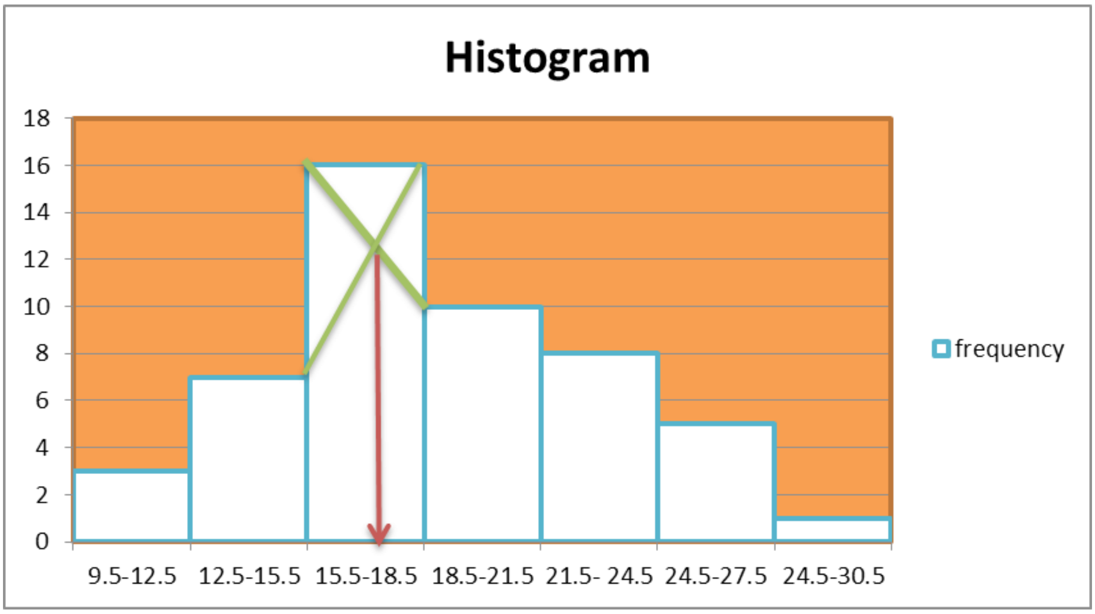
ii) From the top right corner of the modal class to the top right corner of the previous class.
iii) Vertically down from the intersection of the two lines in i) and ii)

The mode is the value where the line (iii) touches the horizontal axis.

For the grouped frequency distribution, the masses of 50 stones on a tipper truck gave the following distribution.

Mass kg	10-12	13-15	16-18	19-21	22 - 24	25-27	28-30
frequency	3	7	16	10	8	5	1

Required: Find the estimate mode



From the histogram the estimate mode is 17

1. **Selection of a measure of central tendency**:

All the three measures of central tendency are designed for the same work - that is to give an indication of the central value of a distribution. Each is obtained using a different method. However, the most important question to consider when deciding which one to use, you must use the one that is most representative of the data in the distribution.

- Type of variable discrete or continuous distribution.
- Shape of the distribution symmetrical, skewed (positively or negatively).
- Single mode, bi-mode or multimode
- Advantages and disadvantages:

Advantages and disadvantages

	Advantages	Disadvantages
Mean	Easy to understand Takes into account all data value Least likely to be affected by sampling errors Straight forward to calculate Can be used for further algebraic calculation Is stable usually does not differ from sample to sample	Badly affected by extreme value Rarely correspond to an actual value in the data It is not a good measure in case of rations. Sometime it may give absurd answers
Median	Easy to calculate Not affected by extreme value Can be calculated when not all values are known	It is not well defined It is not based on all values of a distribution It is not capable for further mathematical treatment Sometimes the data may have more than one median
Mode	Not affected by extreme values It is in most cases an important value of the variable	It is less representative because it does not depend on all values. It is affected by sampling fluctuations. It is not capable of further mathematical treatment.
Mean	Easy to understand Takes into account all data value Least likely to be affected by sampling errors Straight forward to calculate Can be used for further algebraic calculation Is stable usually does not differ from sample to sample	Badly affected by extreme values rarely correspond to an actual value in the data it is not a good measure in case of ratios. Sometime it may give absurd answers.
Median	Easy to calculate Not affected by extreme value Can be calculated when not all values are known	
Mode	Not affected by extreme values It is in most cases an important value of the variable	

9.7. Weighted mean

In certain situations, the mean does not reflect the true picture of the distribution, for example on expenditure on basic necessities in one particular year.

Example 9.19

Items	**Food**	**drink**	**housing**	**transport**	**entertainment**
Present price (Frw '000)	**150**	**130**	**200***	**120**	**180**

The mean present price $=\frac{150+130+200+120+180}{5}=156\frac{150+130+200+120+180}{5}=156$

Calculate the geometric mean

A weighted mean is an average computed by giving different weights to some individual values. If the weights are all equal, and then the weighted mean is the same as arithmetic mean.

Whereas weighted means generally have a similar approach to arithmetic mean, they have distinctive properties.

Data values with a high weighted contribute more to the weighted mean than the data values with lower weights. The weights like frequencies cannot be negatives. Some may have a weight of zero but not all of them, since division by Zero leads to infinity.

Note. Each of the above items affects individuals differently. If an individual does not travel a lot, the price increase in transport will not have the same impact as other price increases. We, therefore, attach a weight or a measure of importance to each item in order to get a more accurate average that applies to an individual.

Example 9.20

We attach a weight to the example above.

Items	**Food**	**Drink**	**Housing**	**Transport**	**Entertainment**
Presentprice(1000Frw	150	130	200	120	180
Weight	8	6	5	2	3

Required :

Calculate the weighted mean.

Weighted mean $=\frac{\sum wx}{\sum w}=\frac{\sum wx}{\sum w}$

Weighted mean $=\frac{8x150+6x130+5x200+2x120+3x180}{8+6+5+2+3}=\frac{8x150+6x130+5x200+2x120+3x180}{8+6+5+2+3}=\frac{3760}{24}\frac{3760}{24}$=156.667

1. **Geometric Mean**

The arithmetic mean is the sum of values in a distribution divided by their number while geometric mean is the anti-logarithms of the values divided by their number.

Geometric mean is defined as the nth root of a product of N items in a series. Therefore, geometric mean (GM)=antilog ($\frac{\sum logx}{N}\frac{\sum logx}{N}$) where x refers to various values distribution.

Example 9.21

The harvests of maize in 10farms in Gatsibo district are given in the table below

Calculate the geographic mean.

X	LogX

7.5	0.8751
13	1.1139
18.5	1.2672
20.5	1.3118
5.2	0.7160
23	1.3617
24	1.3802
25	1.3979
26	1.4150
28	1.4472
	∑x= 12.2860

GM =Antillog $\frac{12.2860}{10}$

= 16.92 (1000s)

Example 9.22
The following are subject passes score by 10 girls in a test.
5, 4. 4, 4, 6, 5,4. 3, 12. 4.

Required:

a) Calculate:
   - the mean; and
   - the geometric mean.

b) Compare the arithmetic mean and geometric mean and establish which measure is more representative of the girl's performance.

Solution:

a)
- Arithmetic mean = 51/10=5.1
- Geometric mean

Product of the numbers = 5529600

Geometric mean tenth root of product of the numbers = $55529600^{\frac{1}{10}}$ $55529600^{\frac{1}{10}}$ =4.724

The geometric mean is much more representative of the distribution than the mean since 8 of 10 girls obtained a lower grade than the mean 5.1.

a) **Merits of geometric mean:**
- Geometric mean is based on each value in the time series.
- Geometric mean is well defined.
- Geometric mean is useful in averaging ratios,
- Percentages and geometric progression series.

b) **Limitations**
- Difficult to interpret.
- Cannot be calculated when their negative values or when one variable in the series is zero.
- Has limited application in further mathematical computations

2. Harmonic mean

The harmonic mean is calculated as the mean of the reciprocals of the numbers or values in a distribution. The harmonic mean is based on all values in the distribution. Harmonic means gives more weight to small values and less weight to large values. Therefore, harmonic mean is used when we need to give greater weight to small items. It is mainly applied in cases of times and average rates. Harmonic mean can be calculated using the formula HM =

a) calculation of harmonic mean

Example 9.23

Find the harmonic mean for the following individual data values:

X	**1/x**
0.01	100
0.5	2
1	1
4	0.25
10	0.1
11.2	0.0893
45	0.0222
175	0.0057
	103.4672

$$\sum \frac{1}{x} = 103.4672$$

The harmonic mean is given by $\frac{N}{\sum\frac{1}{x}}\frac{N}{\sum\frac{1}{x}}$ therefore the harmonic mean of our distribution is $HM = \frac{N}{\sum\frac{1}{x}} = \frac{8}{103.4672} = 0.077 HM = \frac{N}{\sum\frac{1}{x}} = \frac{8}{103.4672} = 0.077$

The harmonic mean can be given by $HM = \frac{N}{\sum f(\frac{1}{x})} HM = \frac{N}{\sum f(\frac{1}{x})}$ consider the example below find the harmonic mean

Value of (x)	1	3	5	7	9	11
Frequency (f)	2	4	6	8	10	12

Solution

Value of x	Frequency f	1/x	f*1/x
1	2	1	2.00
3	4	0.3333333	1.33
5	6	0.2	1.20
7	8	0.1428571	1.14
9	10	0.1111111	1.11
11	12	0.0909091	1.09
Sum	42		7.88

$$HM = \frac{N}{\sum f(\frac{1}{x})} \Rightarrow \frac{42}{7.88} = 5.3$$

Merits of Harmonic Mean:

It well defined and is based on all values.
It is capable of being used for further mathematical treatment

Demerits of harmonic mean

Harmonic mean is not defined when one of the values in the distribution is zero
Harmonic mean is high affected by extreme values

Solved Exercises

Question 9.1

The table shows a university program and the percentage of students who passed the

final examination.

Program	Percentage pass	Number of students 1000's
MA	71	3
MAS COM	83	4
BA	73	5
BCOM	74	2
BSC	65	3
MSC	66	3

Required:

Calculate the weighted mean.

Solution:

The weighted mean = 72,550 students.

Question 9. 2

A candidate scored an A in business English (3 credits), a C in business law (3 credits), a B in commercial environment (4 credits) and a D in financial accounting (credits).
Assuming A = 4 grade point, B = 3 grade points, C = 2 grade points, D = 1 grade point and F = 0 grade point.

Required:

Find the candidates grade-point average.

Solution:

The grade-point average is 2.7 points.

Question 9.3

The coffee yield in 1000s of Kgs from 10 farmers of Tuzamurane Cooperative in one month is shown in the table below

Farmer	1	2	3	4	5	6	7	8	9	10
Yield in 1000s kg	7.5	13	18.5	20.5	5.2	23	24	24	26	28

Required:

Calculate the geometric mean yield. Solution:

The geometric mean yield is 16.92 x 1,000kg

Question 3

The mean of the following frequency distribution is 62.8.

Class	0-20	20-40	40-60	60-80	80-100	100-120	Total
Frequency	5	Fi	10	f2	7	8	50

Find the values of f1 and f2.

Solution:

f 1 = 8 and f2 = 12.

CHAP 10

MEASURES OF DISPERSION

10.1. Study objectives

By the end of this chapter, you should be able to:

- distinguish between the various measures of dispersion and compute their values;
- identify the merits and demerits of each measure of dispersion; and
- compute standard deviation and variance.

10.2. Central tendency versus dispersion

The various measures of central tendency give one single value that represents the entire data. However, average alone cannot adequately describe a set of observations, unless all the observations are alike. It is therefore necessary to describe the variations or dispersion of the observations. In two or more distributions, the measure of central tendency may be the same and yet there can still be disparities in the formation of the distributions.

Consider the following sets of data:

a) 9, 9, 9, 9, 9;
b) 7, 8, 9, 10, 11; and
c) -191,-91, 9, 109, 209.

Although the mean in each case is 9, the last distribution is much more spread out than the other two, thus the need for the measure of spread or dispersion.

Definition 10.1

Dispersion is the degree or extent of spread of items in a distribution around a measure of central tendency. A measure of dispersion indicates the extent to which the individual observations differ from the mean or from any other measure of central tendency.

The measures of dispersion are also called the measure of **variation or measures of spread.** The measure of dispersion is said to be **absolute** if it is expressed in the units of the variables. However, if it is expressed in the form of a co-efficient, ratio or percentage then it is said to be **a relative measure of dispersion.**

Need or purpose for measures of dispersion

The measures of dispersion are determined so as to:

- establish the reliability of the measure of central tendency;
- serve as the basis for the control of the variability;
- compare two or more series of distributions as far as their variability is concerned; and
- Facilitate the use of other statistical measures.

Characteristics of a good measure of dispersion

The desirable properties of a good measure of dispersion include:

- Simple to comprehend;
- easy to compute;
- specifically defined, using all the items in the distribution for its computation;
- easily used in further algebraic calculations;
- unaffected drastically by the extreme items in the distribution; and
- affected as little as possible by fluctuations in sampling.

10.2.1. Measures of dispersion

Some of the measures of dispersion that are commonly used include the following:

- range;
- quartile deviation (inter-quartile range);
- percentile range; standard deviation; and
- Variance etc.

The range

The range is based entirely on the extreme values of the given distribution and it is given by the difference between the highest value and the lowest value of the data; that is:

Range = highest value (HV) - lowest value (LV).

This is the absolute range or merely referred to as just the range.

The relative (coefficient) range on the other hand is the ratio of the absolute range to the sum of the extreme values in the distribution (or any measure of central tendency i.e. mode, mean and median), that is:

Coefficient range $\frac{HV-LV}{HV+LV} orcoefficientrange = \frac{HV-LV\ HV-LV}{averageHV+LV} orcoefficientrange = \frac{HV-LV}{average}$

Note: The average used in determining the co-efficient range represents any measure of central tendency.

Example 10. 1

Given the following distributions:

i) 9. 9, 9, 9, 9;
ii) 7, 8, 9, 10, 11; and
iii) -191,-91,9,109,209,

Find the range in each case.

Solution:

- range = 9 - 9 = 0.
- range = 1 1 - 7 = 4.
- range = 209 - -191 = 400.

Example10.2

The retiment packages of different beneficiaries are given in the table below

Retirement package (in millions	10-20	20-30	30-40	40-50	50-60

No of beneficiary	8	10	12	8	4

Required:

Calculate the:

Range;and
Co-efficient of range.

Solution

The range =upper class boundary of the highest class (UCB) – the lower –class boundary of the lowest class (LUB) =60-10=50.

the co-efficient of range

$\frac{\text{the range the range}}{UCB+LCBUCB+LCB}=\frac{50 \quad 50}{60+1060+10}$

Uses of range in quality control

In the control charts prepared for this purpose, the range plays a key role. It is, therefore, important to note that should the range rise beyond a certain point, the production machines have to undergo routine service.

- Variation of prices:

Changes in prices of stocks and shares and other commodities that are sensitive to rice changes from one period to another, can be monitored using the range.

- Weather forecasts:

Meteorological departments do make use of the range in ascertaining say the difference between the minimum and the maximum temperatures. This information is vital to the general public because they know as to within what limits the temperature is likely to vary on a particular day.

- Daily use:

A wage range is as important to a person seeking employment as the average rainfall over a certain period is to a farmer considering planting his crops.

b) Advantages

- Easy to understand.
- Simple to compute.

Disadvantages

- Does not consider all values of the distribution.
- Cannot be computed when the distribution is open ended.
- Does not give anything about the character of the distribution within the two extreme observations
- Fluctuates a great deal from sample to sample.

Quartile deviation

Quartile deviation is another measure of dispersion. Quartile deviation is also called the semi interquartile range.

Quartile deviation $=\frac{\text{inter quartile range inter quartile range}}{2 \qquad 2}$

The inter quartile range is given by Q_3 – Q1, where Q3 is the upper quartile while Q1 is the lower quartile.

Therefore:

The quartile deviation $=\frac{Q3-Q1Q3-Q1}{2 \quad 2}$

Another measure of dispersion related to the inter quartile range is the quartile coefficient of the dispersion. This is given by:

a) Quartile Coefficient of dispersion = $\frac{Q3-Q1}{median}$X100%$\frac{Q3-Q1}{median}$X100%

Example 10.3

Seven students who sat for an aptitude test in a certain organization scored the following marks:
30, 38, 50, 22, 40, 25, 60

Required:

Find the value of the:

- quartile deviation; and
- quartile coefficient of the dispersion.

Solution:

a) Rearranging the scores:
22, 25, 30, 38, 40, 50, 60.

Position of Q, =$\frac{(n+1)th}{4}\frac{(n+1)th}{4}$=$\frac{7+1}{4}\frac{7+1}{4}$=2 ; i.e., the lower quartile is the 2nd item.

Therefore, Q_1 = 25

Position ofQ_3 = $\frac{3(N+1)th}{4}\frac{3(N+1)th}{4}$=$\frac{3(7+1)th}{4}\frac{3(7+1)th}{4}$

=6th

Q_3=50

Then, the quartile deviation = $\frac{Q3-Q1}{2}=\frac{50-25}{2}\frac{Q3-Q1}{2}=\frac{50-25}{2}$

= 12.5

b) Quartile Coefficient of the dispersion

But the median of 22, 25, 30, 38, 40, 50. 60 is 38

Then the Quartile Coefficient of the dispersion = $\frac{Q3-Q1}{median}$X100%$\frac{Q3-Q1}{median}$X100%

$\frac{50-25}{38}$X100%

=65.79%

Mean deviation

Definition10.2 Mean deviation is the measure of dispersion that gives the average absolute difference between each item and the mean.

It should be **noted** that the mean deviation is a much more representative measure than the range since **all** item values are considered in its calculation.

The mean deviation is obtained from a suitable formula, among the following:

For a complete set of data, **Mean deviation =** $\sum\frac{|x-\bar{x}|}{n}\sum\frac{|x-\bar{x}|}{n}$;

For a frequency distribution, **Mean deviation =** $\sum f\frac{|x-\bar{x}|}{\sum f}\sum f\frac{|x-\bar{x}|}{\sum f}$

Note: The absolute difference, |x - x\means just the magnitude/size of the difference and ignores the negative sign.

Example 10.4

The ages of eight people in a certain village are:
23, 55, 28,19,31,27, 30, 27.

Required:

Calculate the mean deviation of the ages.

Solution

The mean age, $\bar{x}\bar{x}=\frac{23+55+28+19+31+27+30+27}{8}\frac{23+55+28+19+31+27+30+27}{8}=\frac{240}{8}\frac{240}{8}=30$

Then the mean deviation $\sum\frac{|x-\bar{x}|}{n}\sum\frac{|x-\bar{x}|}{n}$

Table:

X	$\lvert x - \overline{xx}\rvert$
23	7
55	25
28	2
19	11
31	1
27	3
Total	49

So mean deviation $=\frac{4949}{8\ \ 8}=6.125$

Example 10.5

The data in the table below captures the deposits (in millions of Rwandan Francs) made in a certain bank in Kigali in week.

Amount deposited	10-14	15-19	20-24	25-29	30-34	35-39
Number of customers	1	14	23	21	15	6

Required:

Calculate the mean deviation of the deposits.

Solution:

Deposit (‘000,000)	No. of customers	Midpoint, x	fx	$\lvert x - \overline{xx}\rvert$	$f\lvert x - \overline{xx}\rvert$
10-14	1	12	12	13.3	13.3
15-19	14	17	238	8.3	116.2
20-24	23	22	506	2.3	52.9
25-29	21	27	567	1.7	35.7
30-34	15	32	480	6.7	100.5
35-39	6	37	222	11.7	70.2
Totals	80		2025		388.8

The mean deposit $=\frac{20252025}{80\ \ 80}=25.3$

Therefore , the mean deviation $=\sum f\frac{|x-\bar{x}|}{\sum f}=\frac{388.8}{80}=4.86\sum f\frac{|x-\bar{x}|}{\sum f}=\frac{388.8}{80}=4.86$ **Million**

Example10.6
The results from a weekly test are given in the table below:

Scores x	Number of candidates f candidates f
10-15	2
15-20	12
20-25	27
25-30	41
30-35	30
35-40	7

Calculate the mean deviation from then mean of this distribution

Solution

Mean$\bar{x} = \frac{\sum fx}{\sum f}\bar{x} = \frac{\sum fx}{\sum f}$= $\frac{3207.5}{119}$ =4.41

X	Mid - value	f	$\lvert x-\bar{x}\rvert$	$f\lvert x-\bar{x}\rvert\lvert x-\bar{x}\rvert$
10-	12.5	2	14.45	28.90
15-	17.5	12	9.45	113.40
20-	22.5	27	4.45	120.15
25-	27.5	41	0.55	22.55
30-	32.5	30	5.55	166.50
35-40	37.5	7	10.55	73.85
		f=119		$\sum f\lvert x-\bar{x}\rvert\sum f\lvert x-\bar{x}\rvert$ =525.35

Mean deviation $\frac{\sum f\lvert x-\bar{x}\rvert}{\sum f}\frac{\sum f\lvert x-\bar{x}\rvert}{\sum f}$= $\frac{525.35}{119}$ =4.41

Characteristics of mean deviation

- It is a good representative measure of dispersion that is easy to understand. It is, therefore, useful for comparing the variability between distributions of like nature.
- The modulus sign makes it impossible to handle the mean deviation

Theoretically and this limits its applicability for advanced analysis.

- When the mean is not a whole number, its computation is rather complicated.

Merits

- It is a good representative measure of dispersion that is not difficult to understand.
- It is useful for comparing the variability between distributions of like nature

Demerits

- Information is lost by ignoring the signs of the deviations, making it difficult to treat it theoretically
- It can be complicated and awkward to calculate if the mean is anything other than a whole number.
- Because of the modulus sign, the mean deviation is virtually impossible to handle theoretically and

thus is not used in more advanced analysis

Standard deviation

Definition 10.3

Standard deviation is the square root of the arithmetic mean of the squares of the deviations measured from the mean.

Other measures of spread based on standard deviation are the coefficient of variation and Pearson's measure of skewness.

Computation of standard deviation

Standard deviation for a set of ungrouped data:

This is given by the formula:

$$\delta = \sqrt{\frac{(x-\bar{x})^2}{n}} \; or \; \delta = \sqrt{(\frac{\sum x^2}{n} - \bar{x}^2)}$$

Example 10.7

Paska noted that on a certain week the total weight (in kg) of the items bought on her daily shopping were as follows:

16, 10, 15, 11,8, 12, 10.

Required:

Find the standard deviation

Solution: It should be noted that a suitable table eases the implementation of the formula and it should be adopted, as illustrated below:

X	x-$\bar{x}\bar{x}$	(x-$\bar{x}\bar{x}$)^2
16	4.2857	18.3672
10	-1.7143	2.9388
15	3.2857	10.7958
11	-0.7143	0.5102
8	-3.7143	13.7960
12	0.2857	0.0816
10	-1.7143	2.9388
∑x = 82		∑(x-$\bar{x}\bar{x}$)^2= 49.4284

The mean $\bar{x}\bar{x} = \frac{\sum x \sum x}{n \;\; n}$=82/7=11.7143

Then the standard deviation $\sqrt{\frac{\sum(x-\bar{x})2}{n}}\sqrt{\frac{\sum(x-\bar{x})2}{n}} = \sqrt{\frac{49.4284}{7}}\sqrt{\frac{49.4284}{7}}$=2.6573

Example 10.8

A convenient computational formula can be deduced from one used above for a set of data and has the form

The number of vehicles sold from a certain bond in a period often months was: 34, 28, 45. 57, 63, 44. 70, 39, 52, 66.

Required:

Calculate the:

mean; and

a) standard deviation

Solution:

X	34	28	45	57	63	44	70	39	52	66	Jx = 498
X^2	1156	784	2025	3249	3969	1936	4900	1521	2704	4356	Xx^2 =26600

a) The mean $\overline{xx} = \frac{\sum x \sum x}{n\ \ n} = \frac{498}{10} = 49.8$

b) Standard deviation S = $\sqrt{\frac{\sum x^2}{n}} = \sqrt{\frac{\sum x^2}{n}} - \overline{x^2 x^2}$

$\sqrt{\frac{26.600}{10}}\sqrt{\frac{26.600}{10}} - 49.8^2 49.8^2 = 13.415$

Standard deviation for a frequency distribution:

The formula used to compute standard deviation for frequency distribution is given by:

$$\sqrt{\frac{\sum fx^2}{n} - (\frac{\sum x^2}{\sum f})2}$$

It should be noted that when handling a grouped frequency distribution, x is the class midpoint.

Example 10.9

The annual salary structure for workers in a certain NGO follows the distribution below.

Salary (Frw millions)	10-14	15-19	20-24	25-29	30-34	35-39
Number of employees	1	14	23	21	15	6

Required:

Calculate the standard deviation.

Solution

Salary_	Midpoint x	Frequency f	Fx	fx^2
10-14	12	1	12	144
15-19	17	14	238	4046
20-24	22	23	506	11132
25-29	27	21	567	15309
30-34	32	16	512	16384
35-39	37	6	222	8214
		$\sum f \sum f$=81	**$\sum X$** f=2057	**$\sum X$** £fx^2=55229

Then the standard deviation s= $\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum x^2}{\sum f}\right)} = \sqrt{\frac{55229}{81} - \left(\frac{2057^2}{81}\right)}\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum x^2}{\sum f}\right)} = \sqrt{\frac{55229}{81} - \left(\frac{2057^2}{81}\right)}$ =6.077

Age x	Number of patients f

10-20	18
20-30	34
30-40	58
40-50	42
50-60	24
60-70	10
70-80	6
80-90	8

Required:

Calculated the variance and standard deviation using working mean method

$$Standard\,deviation = \sqrt{Variance} = \sqrt{282.51} = 16.8$$

Properties of standard deviation and coefficient of variation:

- Standard deviation is only used to measure spread or dispersion around the mean of a data set.
- Standard deviation is never negative.
- Standard deviation is sensitive to outliers. A single outlier can raise the standard deviation and in turn, distort the picture of spread.
- For data with approximately the same mean, the greater the spread, the greater the standard deviation.
- If all values of data set are the same, the standard deviation is zero (because each value is equal to the mean).
- When the mean value is near zero the Coefficient of Variation is sensitive to small changes in the mean.
- The Coefficient of Variation is independent of change of scale but not of origin.

Interpretation of calculated values

- **Variance:** Is a measure of the spread of the distribution about the mean.
- **Standard deviation:** A large standard deviation indicates that the data points are far from the mean and a small standard deviation indicates that they are clustered closely around the mean.

For example, each of the three populations (0,0,14,14), (0,6,8,14) and (6,6,8,8) has a mean of 7 and their standard deviation are 7, 5 and 1 respectively. The third population has a much smaller standard deviation than the other two because its values are all close to 7.

Merits of standard deviation

- It is rigidly defined and its value is always definite and based on all the Observations and the actual signs of deviations are used
- As it is based on arithmetic mean, it has all the merits of arithmetic mean.
- It is the most important and widely used measure of dispersion.
- It is possible for further algebraic treatment.
- It is less affected by the fluctuations of sampling and hence stable.
- It is the basis for measuring the coefficient of correlation and sampling

Demerits of standard deviation

- It is not easy to understand and it is difficult to calculate.
- It gives more weight to extreme values because the values are squared up.
- As it is an absolute measure of variability, it cannot be used for the purpose **of** Comparison.

Determination of quartiles:

a) **Ungrouped data**

If a data set of values is arranged in ascending order of magnitude, then:

- The median is the middle value of the data set. Therefore, median = $\frac{11}{22}$ (n + l) th value where n is the number of data values in the data set.
- The lower quartile is the median of the lower half of the data set. Therefore,

  Q_1= $\frac{11}{44}$ (n + l) th value where n **is** the number of data values in the dataset.
- The Upper quartile (Q_3) is the median of the upper half of the data set. Therefore,

  Q_3 = $\frac{33}{44}$(n + l) th value where n is the number of data value in the data set.

Example 10.10

The following are a set of marks scored by a student in different papers at the end of level one examination:

43, 75, 48, 51, 54, 47, 50

Required:

Calculate the quartiles for the above set of marks.

Solution:

The quartiles of the marks 43, 75, 48, 51, 54, 47, 50 are found by first arranging in ascending order as:

43, 47, 48, 50, 51, 54, 75

Then

$Q_1$is the value of the $(\frac{7+1}{4})^{th}(\frac{7+1}{4})^{th}$ = 2^{nd}item, which is 47;

Q_2 is the value of the $(\frac{7+1}{2})^{th}(\frac{7+1}{2})^{th}$=$4^{th}$ item which is 50; and

Q_3 is the value of the $\frac{33}{44}$ $(7+1)^{th}$ =6^{th} item which is 51.

b) **Grouped data**

If a data set of values is:

- The median is the middle value of the data set. Therefore, median **Q_2 = Lm +-c($\frac{\frac{1}{2}n-F\frac{1}{2}n-F}{fm\quad fm}$)** value where n is the number of data value interval F is the cumulative frequency of the class before the median class fm is the frequency of the median class

- The lower quartile **Q_1 = Lm + c($\frac{\frac{1}{4}n-F\frac{1}{4}n-F}{fm\quad fm}$)**value where n is the number of data

Values in the data set, Lm is the lower limit of the lower quartile class, c is the class interval, **F** is the cumulative frequency of the class before the lower quartile class, fm is the frequency of the lower quartile class

- The

- Upper quartiles **Q_3 = Lm +c($\frac{\frac{3}{4}n-F\frac{3}{4}n-F}{fm\quad fm}$)** value where n is the number of data values in the data set, Lm is the lower limit of the upper class, c is the class interval, F is the cumulative frequency of the class before the upper class, fm is the frequency of the upper quartile class.

Example 10.11

The frequency distribution below shows rural schools' termly expenditure on water during a survey in central Rwanda .

Amount (Frw '000)	**Frequency**
20-24	9
25-29	17
30-34	21
35-39	18
40-44	15
45-49	9
50-54	11

Required:

Computer the quartile, that is Q_1, Q_2 and Q_3 of the water expenditure.

Clas interval	f	Cf	**Class limits**
20-24	9	9	19.5-24.5
25-29	17	26	24.5 - 29.5
30-34	21	47	29.5 - 34.5
35-39	18	65	34.5 - 39.5
40-44	15	80	39.5 - 44.5
45-49	9	89	44.5 - 49.5
50-54	11	100	49.5 - 54.5
	$\sum f \sum f=100$		

To identify the quartile classes, divide the cumulative frequency by

- $\frac{11}{44}$ (25% of data), $\frac{11}{22}$ (50% of data) and $\frac{33}{44}$ (75% of data)
- Class interval refers to the number of elements in a given class, that is, 5 - 9 means there are Five elements from 5 to 9
- Lower limit of is got by reducing the left hand digit by 0.5 Upper limit by increasing the Right hand digit by 0.5. These became the class limits.ie Class interval of 10 - 19 will have class limits as 9.5 - 19.5. Obtain the cumulative frequency

Solution:

- The first quartile class is $\frac{100100}{4\ \ 4}$=25the position which is 25 - 29 because from the table 25 is within 26 in the cumulative frequency column. Therefore, Lm is 24.5, c = 5, $\frac{100100}{4\ \ 4}$ = 25, cfb 9 and fm = 17

Q_1 =Lm+c($\frac{\frac{1}{2}n-F\frac{1}{2}n-F}{fm\ \ fm}$)=24,5+5($\frac{25-925-9}{17\ \ 17}$)25.44

- The second quartile class is $\frac{100100}{4\quad 4}$=50*position* $_{wr1}$jch is 35 - 39 because from the table 50 is within 65 in the cumulative frequency column. Therefore, Lm is 34.5, c = 5,$\frac{100}{4}$,$\frac{100}{4}$= 50,cfb = 47 and fm = 18

$$Q_{2=} Lm+c(\frac{\frac{1}{2}n-F\frac{1}{2}n-F}{fm \quad fm}) \; =34.5+5(\frac{50-4750-47}{18 \quad 18})=34.5+0.167=34.667$$

- The third quartile class is = 75 th position which is 40- 44 because from the table 50 is within 80 in the cumulative frequency column. Therefore, **Lm** is
39.5, c = 5, =p = 75, cf_b=65 and fm=15

$$\mathbf{Q3 = Lm + c}\left(\frac{\frac{3}{4}n-F\frac{3}{4}n-F}{fm \quad fm}\right)$$

Graphical method

- Form the **cumulative frequency distribution.** The cumulative frequency for any class is the sum of frequency of that class and lower classes.
- Plot each cumulative frequency against the upper boundary of the corresponding class interval. 3x100
- Join these points with a smooth curve to form the **cumulative frequency curve** (or ogive) Q_3 = Lm + c = 39.5 + 5
- The middle number of the distribution is located on the cumulative frequency axis and the corresponding value of the variable is the median. The quartiles can be found in a similar way. 39.5 + 0.67 = 40.17

Merits of quartile deviation

- It is rigidly defined
- It is superior to range as such its calculation is based on middle 50% of the items of the series.
- It is easy to calculate especially in case of open end series.
- It is not very much affected by the extreme values of the series
- In a moderately systematic series, it enables the computation of lower quartile, upper quartile, standard deviation and mean deviation.

Demerits of quartile deviation

- It is not based on all the observations of the series
- It is not capable of further algebraic treatment
- It is affected by the fluctuations in sampling
- It is not understood by a common man It is more time consuming.
-

Deciles

A decile is a fraction relating to a tenth of a distribution.

Calculating Deciles

- Order the data from smallest to largest.
- Find the position that occupies every decile using the expression$\frac{k.N^{th}k.N^{th}}{10 \quad 10}$, where k = 1,2………9 and N is the total number of items in the distribution

- The kth decile D_k.is calculated using the formula D_k=Li+$\frac{\frac{kn}{10}-Fi-1}{fi}xci\frac{\frac{kn}{10}-Fi-1}{fi}$xci

where

k = 1,2,…….9

L_i= the lower boundary of the decile class

N = the total frequency

F_{i-1}= the cumulative frequency below the decile class4th

Ci= the class width of the decile class

fi =the frequency of the decile class

Example 10.14

Table below shows the weights of patients in a referral hospital.

Weights of patients (kg)	**fi**
50-60	8
60-70	10
70-80	16
80-90	14
90-100	10
100-110	5
110-120	2

Required:

Calculate the deciles of the distribution.

Solution:

Weight of patients	**fi**	**Fi**
50-60	8	8
60-70	10	18
70-80	16	34
80-90	14	48
90-100	10	58
1C0- 110	5	63
110-120	2	65
	65	

Use $\frac{kN^{th}kN^{th}}{10\quad 10}$ to find the position of the deciles and to calculate the deciles

For D_1 from $\frac{kN^{th}}{10}$, $k = 1, N = 65$ postion of decile class is $\frac{65x1kN^{th}}{10\quad 10}$, $k = 1, N = 65$ postion of decile class is $\frac{65x1}{10}$ =6.5th

This position lies in the class 50-60 (first decile class)

Then L_1=50,C1=10 f1=8 and Fo=0

Substitution in the formula, D_k=Li+$\frac{\frac{kn}{10}-Fi-1}{fi}xci\frac{\frac{kn}{10}-Fi-1}{fi}$xci

D1 =50+$\frac{6.5-06.5-0}{8 \quad 8}$x 10 = 58.1210 = 58.12

Similarly D2 to D9 are computed as below:

For D2, $\frac{65x265x2}{10 \quad 10}$=13 D2= $60 + \frac{13-8}{10}$x10 $60 + \frac{13-8}{10}$x10=65

D3 =$\frac{65x365x3}{10 \quad 10}$=19.5 D3=70 + $\frac{19.5-18}{10}$x1070 + $\frac{19.5-18}{10}$x10=70.94

D4=$\frac{65x465x4}{10 \quad 10}$=26 D4=70 + $\frac{29-18}{10}$x1070 + $\frac{29-18}{10}$x10=75

D5=$\frac{65x565x5}{10 \quad 10}$=32.5 D4=70 + $\frac{32.5-18}{10}$x1070 + $\frac{32.5-18}{10}$x10=79.06

D6=$\frac{65x665x6}{10 \quad 10}$=39 D4=70 + $\frac{39-34}{10}$x1070 + $\frac{39-34}{10}$x10=83.57

D7=$\frac{65x765x7}{10 \quad 10}$=75.5 D4=70 + $\frac{45.5-34}{10}$x1070 + $\frac{45.5-34}{10}$x10=88.21

D8=$\frac{65x865x8}{10 \quad 10}$=52 D4=70 + $\frac{52-48}{10}$x1070 + $\frac{52-48}{10}$x10=94

D9=$\frac{65x965x9}{10 \quad 10}$=58.5 D4=70 + $\frac{58.5-58}{10}$x1070 + $\frac{58.5-58}{10}$x10=101

Percentiles

Percentile rank of an observation is the percentage of observations in the entire distribution with similar or smaller values than that observation.

For instance, a height has a percentile rank of 90 if equal or shorter heights constitute 90% of the entire distribution.

Percentile rank = $\frac{individual\ cummulative\ frequency individual\ cummulative\ frequency}{total\ frequency \qquad total\ frequency} x10$

Example 10.15

The table below shows marks of 50 students in a QT pre-test.

Marks	Frequency
52	9
57	7
62	11
67	8
72	5
77	9
82	1

Required:

Determine the percentile rank of the distribution.

Solution:

Marks	Frequency	Cumulative frequency	I Percentile ranks
52	9	9	18
57	7	16	32
62	11	27	54

67	8	35	70
72	5	40	80
77	9	49	98
82	1	50	100

Example 10.16
Percentile rank of a grouped data

To find the percentile rank of a score **x** out of n scores where **x** is not include:

Percentile rank = $\frac{\text{number of scores below}}{n}x100\frac{\text{number of scores below}}{n}x100$

Example 10.17
Byamukama is position 30th in a class out 140 students.

Required:

Find his percentile rank in the class

Solution:

Using Percentile rank $\frac{\text{number of scores below}}{n}x100\frac{\text{number of scores below}}{n}x100$

110 students are ranked below Byamukama.

Byamukama's percentile rank would be $\frac{110\ 110}{140\ 140}$ x 100 = 79th percentile.

Example 10.18
The QT test scores were: 50, 65, 70, 72, 72, 78, 80, 82, 84, 84, 85, 86, 88, 88, 90, 94, 96, 98, 98, 99.
Required:
Find the percentile rank for a score of 84 on this list.

Solution:
The scores must be ordered from smallest to largest. Locate the 84.
Since there are 2 values equal to 84, assign one to the group "above 84" and the other to the group "below 84".
50, 65, 70, 72, 72, 78, 80, 82, 84, 84, 85, 86, 88, 88, 90, 94, 96, 98, 98, 99
Percentile rank $\frac{\text{number of scores below}}{n}x100\frac{\text{number of scores below}}{n}x100$

$=\frac{9\ \ 9}{2020}$x100=45th percentile.

The score of 84 is at the 45,th percentile for this test.
Interpretation of Percentiles:
Percentiles are useful for giving the relative standing of an individual in a group. Percentiles are essentially normalised ranks. The 80th percentile is a value where you'll find 80% of the values lower and 20% of the values higher. Percentiles are expressed in the same units as the data.
Quartile coefficient of dispersion (qcd)
(qcd) = $\frac{\text{number of scores below}}{n}x100\%\frac{\text{number of scores below}}{n}x100\%$

Example 10.19

The frequency distribution below shows Schools termly expenditure in thousands of Rwandan Francs on water during a survey in Central Rwanda .

Amount ('000)	20-24	25-29	30-34	35-39	40-44	45-49	50-54
Frequency	9	17	21	18	15	11	9

Required:

Compute the i) quartile deviation ii) quartile coefficient of dispersion.

Solution:

a) Quartile deviation $\frac{Q3-Q1}{2}\frac{Q3-Q1}{2}$

Q1=Lq$_1$+c($\frac{\frac{1}{4}n-cfb}{fq1}) = 24,5+5(\frac{25-9}{17}) = 25.44\frac{\frac{1}{4}n-cfb}{fq1}) = 24,5+5(\frac{25-9}{17}) = 25.44$

Q2= Lq$_3$+c($\frac{\frac{3}{4}n-cfb}{fq3}) = 39.5+5(\frac{75-69}{15}) = 40.17\frac{\frac{3}{4}n-cfb}{fq3}) = 39.5+5(\frac{75-69}{15}) = 40.17$

Therefore quartile deviation $\frac{40.17-25{,}44}{2} = 14{,}73\frac{40.17-25{,}44}{2} = 14{,}73$

I) **qcb** $\frac{quartiledeviation}{medium}100\% = 14{,}73\frac{quartiledeviation}{medium}100\% = 14{,}73$

But median = **Lm+c(**$\frac{\frac{1}{2}n-cfb}{fm}) = \frac{\frac{1}{2}n-cfb}{fm}) =$

34.5+5($\frac{50-47}{18}) = 34.667\frac{50-47}{18}) = 34.667$

qcb=42.49%

Self-test questions

Question 10.1

Define the following measures of dispersion:

a) Range.

b) Quartile deviation,

c) Mean deviation,

d) Standard deviation.

Solution:

a) The definition of range can be found in Chapter 7, section 5a).

b) The quartile deviation definition can be found in Chapter 7, section 6.

c) The mean definition can be found in Chapter 7, section 7.

d) The definition of standard deviation can be found in Chapter 7, section 8

Question 10.2

Determine the range.

Solution: The range is 6.2 sales

Question 10.3

The table below gives the frequency distribution of the marks of students who passed an examination.

Sales	Number of companies
10.5-11.6	4
11.7-12.8	6
12.9-13.0	8
13.1 - 14.2	12
14.3-15.4	9
15.5-16.6	8
16.7-17.8	3

The table below gives the frequency distribution of the marks of students who passed an examination.

Marks in %	30-	40 -	50-	55-	60-J	70-
Frequency	0	20	30	35	10	0

Required:

Calculate the mean mark and standard deviation.

Solution:

The mean= 54.1% and the standard deviation = 5.9%.

Question 10.4

The following are the advertising expenditure for a number of companies in a month:

Expenditure in Frw '000	No of companies
Less than 500	210
500 – 10000	184
1000 – 1500	232
1500 -2000	348
2000 – 2500	177
2500 – 3000	83
3000 – 3500	48
3500-4000	12
4000 and above	9

Require

Estimate

- Median = 1500.
- Quartile deviation = 600.
- The quartile coefficient of dispersion = 40%.

Question 10.5

Students of Accounts College obtained the following marks out of 100 in their first test at the college.

Serial No.	1	2	3	4	5	6	7 :	8	9	10
Marks %	5	10	20	25	40	42	45	48	70	80

Required:

Calculate the students' standard deviation

Solution: The standard deviation is 23.07%.

Question 10.6

The following table is an inclusive class frequency distribution

Class boundaries	19.5 – 29.5	29.5 – 39.5	39.5 – 49.5	49.5 – 59.5	59.5 – 69.5	69.5 – 79.5	79.5 – 89.5	89.5 – 99.5
Frequency	5	12	15	20	18	10	6	4

Required:

Calculate the variance and standard deviation.

Solution:

Variance = 311.56 and standard deviation = 17.65.

CHAP 11.

MEASURES OF SKEWNESS

11.1. Study objectives

By the end of this chapter, you should be able to:

- explain the concepts skewness and kurtosis;
- determine relationship among mean, mode and median in a skewed distribution:
- illustrate skewness and kurtosis graphically; and
- interpret skewness and the degree of skewness.

1. Relationship between mean, median and mode

When the data are distributed normally, the three measures of central tendency all fall at or near the same central value of a normal distribution. But when data are skewed, these measures are no longer identical. The figure shows the relationship among the three measures of a positively skewed distribution.

Skewed distribution is neither symmetric nor normal because the data values trail off more sharply on one side than on the other. In business, skewness is often found in data sets that represent sizes using positive numbers, for example sales, assets, customers (population). This is because data values cannot be less than zero (imposing a boundary on one side) but are not restricted by a definite upper boundary. The result is that there are lots of data values concentrated near zero, and they become systematically fewer and fewer as you move to the right in the histogram.

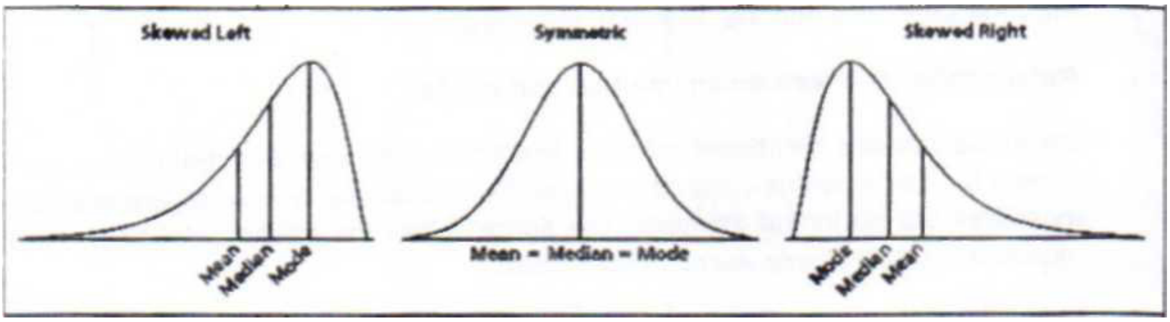
11.2. Skewness

The tendency of the distribution to "tail off to the right or left is called skewness. If the histogram of a grouped frequency distribution is drawn, it usually displays quite low frequencies on the left, builds steadily up to a peak and then drops steadily down to low frequencies again on the right. If the peak is in the centre of the histogram and the slopes on either side are virtually equal to each other, the distribution is said to be symmetrical as shown in figure 2.8 b). If the peak lies to either side of the centre of the histogram, the distribution is said to be skewed. The degree of skewness may be judged by looking at the histogram or by comparing the mean and median. The skewness of a distribution can be measured as regards the:

a) direction of the skewness: and
b) degree of skewness.

11.2.1. Direction of the skewness

The direction of skewness depends on the relationship of the peak to the centre of the histogram, and is indicated by the terms positive skew and negative skew. The skew is positive when the peak lies to the left of the centre (figure c)) and negative when the peak lies to the right of the centre (figure a)).



When distributions are skewed, the median generally lies between the mode and the mean, and the following relationship is satisfied.

(mean - mode) = 3(mean - median)

- If mean > mode, the skew is positive
- If mean < mode, the skew is negative
- If mean = mode, the skew is zero and the distribution is symmetrical

11.2.2. Calculation of skewness

Figure re **(a)** **(b)** **(c)**

Degree of skewness

There are many ways of measuring the degree of skewness. In this topic, we shall illustrate two methods:

i) Karl Pearson's co-efficient of skewness; and
ii) the Bowleys co-efficient of skewness.
iii) The Karl Pearson co-efficient is based on the mean, mode and standard deviation,

Karl Pearson's method (SK)

The Karl Pearson's Method (SK) $= \frac{\bar{x}-mo\bar{x}-mo}{s \quad s}$

The calculated value will lie between -1 and +1.

When a distribution is positively skewed, the value of SK will be positive when the distribution is negatively skewed the value of SK will be negative. But when the mode is not defined. Karl Pearson method becomes

SK $= \frac{3(\bar{x}-m0)3(\bar{x}-m0)}{s \quad s}$

Where $\overline{XX}$ = mean

m_d = median

S = standard deviation

Generally, the values of skewness will range from -3 to +3. The higher the coefficient the, greater the skewness. If the distribution is not skewed at all (but symmetrical), the coefficient will be zero.

Example 11.1

Given the frequency distribution

x	6	18	24	30	36	42
Frequence	4	7	9	18	13	5

Required:

a) Determine the mode.
b) Determine the mean.
c) Determine the standard deviation
d) Hence, find person's co-efficient of skewness.

Solution:

X	f	d = x-30	d^2	fd	fd^2
6	4	-24	576	-96	2304
12	7	-18	324	-126	2288
18	9	-12	144	-108	1296
24	18	-6	36	-108	648
30	15	0	0	0	0
36	10	6	36	60	360
42	6	12	144	60	720
	68			-318	7596

a) $\overline{xx}=30-\frac{318318}{68\ 68}=25.324$

b) Mo=24

c) $S = \sqrt{\frac{\Sigma fd^2}{n} - (\frac{\Sigma d^2}{n})^2}\sqrt{\frac{\Sigma fd^2}{n} - (\frac{\Sigma d^2}{n})^2}$

$S= \sqrt{\frac{7596}{68} - (\frac{-318}{68})^2}\sqrt{\frac{7596}{68} - (\frac{-318}{68})^2}$

$= \sqrt{89.8365}\sqrt{89.8365} = 9.4782$

e) SK = $\frac{\overline{x-mox-mo}}{s\ \ s}$=24.324-4/9.4782 =+0.1397 which show positive skewness
= + 0.1397 which show positive skewness

Example 11.2

In a certain organisation, brief meetings are held daily at 8.00am and the policy is that no meeting should take more than 1 hour. From records a sample of 10 such meetings are taken whose duration in minutes are: 55.7, 53.2, 56.8, 53.8, 52.7, 59.3, 54.2, 53.6, 54.0, 53.7.

Required:

a) Determine the median duration of the meetings.
b) Calculate the mean duration of the meetings.
c) Calculate the standard deviation of the meetings.
d) Calculate the Pearson's coefficient of skewness.

Use the formula, Pearson's coefficient of skewness = $\frac{3(mean-median)\quad 3(mean-median)}{s\tan dard\ dedviation s\tan dard\ dedviation}$

Solution :

a) Median is the middle value after all values are arranged in order of size. Rearranging the durations in order of size:

52.7, 53.2, 53.6, 53.7, 53.8, 54.0, 54.2, 55.7, 56.8. 59.3 The median is halfway between 53.8 and 54.0 That is, median = !4 (53.8 +54.0) = 53.9 minutes

b) Mean duration of meetings

$$=\frac{52.7 + 53.2 + 53.6 + 53.7 + 53.8 + 54.0 + 54.2 + 55.7 + 56.8 + 59.3}{10}=\frac{547}{10}=54.7\text{Min}$$

b) Standard deviation of time of the meetings

Time (min.)	**X -** 54.7	(x - 54.7)^2
52.7	-2.0	4.000
53.2	-1.5	2.250
53.6	-1.1	1.210
53.7	-1.0	1.000
53.8	-0.9	0.810
54.0	-0.7	0.490
54.2	-0.5	0.250
55.7	1.0	1.000
56.8	2.1	4.410
59.3	4.6	21.16
		$\sum(x-54.7)^2=36.58$

SD= $\sqrt{\frac{\sum(x-54.7)^2}{n-1}}\sqrt{\frac{\sum(x-54.7)^2}{n-1}} = \sqrt{\frac{36.58}{9}}\sqrt{\frac{36.58}{9}}$=2.016

d)Pearson's coefficient of skewness = $\frac{3(mean-medium)\ 3(mean-medium)}{standard\ deviation\ standard\ deviation}$

$=\frac{3(54.7-53.9)3(54.7-53.9)}{2.016 \quad 2.016}$

=1.1905

Example 11.3

The following table shows marks of 150 candidates in financial accounting paper.

Marks	0-9	10-19	20-29	30-39	40-49	50-59	60-69	71-79
Frequency	10	40	20	0	10	40	16	14

Using formula

SK = $\frac{3(\frac{\bar{X}}{S}-md)3(\frac{\bar{X}}{S}-md)}{s \quad s}$

Compute the skewness of the distribution

Marks	f	X	d = x-A	d^2	fd	fd^2
0-9	10	4.5	-30	900	-300	9000
10-19	40	14.5	-20	400	-800	16,000
20-29	20	24.5	-10	100	-200	2000
30-39	0	34.5	0	0	0	0
40-49	10	44.5	10	100	100	1000
50-59	40	54.5	20	400	800	16,000
60-69	16	64.5	30	900	480	14,400
70-79	14	74.5	40	1600	560	72,400
	150				**640**	**80800**

$\overline{XX}$=A+$\frac{efdefd}{n\quad n}$ 34.5+$\frac{640640}{150150}$=38.77

S= $\sqrt{\frac{80800-}{150}}(\frac{640}{150})^2\sqrt{\frac{80800-}{150}}(\frac{640}{150})^2$=22.82

X	*f*	Cf
4.5	10	10
14.5	40	50
24.5	20	70
34.5	0	70
44.5	10	80
54.5	40	120
64.5	16	136
74.5	14	150

Since the distribution is a bimodal we do not calculate the mode.

Median =$\frac{n}{2}=\frac{150}{2}=75\frac{n}{2}=\frac{150}{2}=75$

Median = L+$\frac{(\frac{N}{2}-cfb)xc(\frac{N}{2}-cfb)xc}{fm\quad fm}$

=44.5+$\frac{75-8075-80}{40\quad 40}$x10=44.5

=44.5-15.5=43.25

SK = $\frac{3(\bar{x}-md)3(\bar{x}-md)}{s\quad s}$

=$\frac{3(38.77-43.25)3(38.77-43.25)}{22.82\quad 22.82}$=0.589

The negative sign indicates negative skewness

Example 11.4

The following record was found in the records of a human resource manager relating to wages of workers.

- Arithmetic mean - Frw 56, 800
- Median - Frw 59,500
- Standard deviation - Frw 12.400

Required:

i) Calculate coefficient of variation

ii) Determine the skewness of the records

Solution:

i) Coefficient of variation $= \frac{\alpha\alpha}{xx}$ x 100

12400 56800

x 100 = 21.83

ii) SK = $\frac{3(\bar{x}-md)3(\bar{x}-md)}{s \quad s}$

$= \frac{3(56{,}800-59500)3(56{,}800-59500)}{12400 \quad 12400}$ = -0.653

Negative sign indicates negative skewness. This means majority of the workers earn more than mean.

b) Bowley's co-efficient of skewness

This method of skewness is based on quartiles. Lower quartile and upper quartile Q_1 and Q_2 respectively and the median.

Bowley's co-efficient of skewness is given by:

SK = $\frac{Q3+Q1-2mdQ3+Q1-2md}{Q3-Q1 \quad Q3-Q1}$

Example 11.5

The following figures represent a distribution for starting companies in million Rwandan Francs .

Capital (1,000,000sh)	Number of companies
1-5	20
6-10	27
11-15	29
16-20	38
21-25	48
26-30	53
31-35	70

Required:

Determine the lower median and upper quartier, hence, determine whether the distribution is skewed

Solution:

Capital (Frw1,000,000) X	**Class boundaries**	**Number of companies (f)**	**Cf**
1-5	0.5-5.5	20	20
6-10	5.5-10.5	27	47
11-15	10.5-15.5	29	76
16-20	15.5-20.5	38	114
21-25	20.5-25.5	48	162
26-30	25.5-30.5	53	215
31-35	30.5-35.5	70	285
		285	

Median = L+$\frac{(\frac{N}{2}-cfb)xc(\frac{N}{2}-cfb)xc}{fm \qquad fm}$

=20..5+$\frac{142.5-114142.5-114}{48 \qquad 48}$x5=23.4688 millions

Q1 = L+$\frac{(\frac{N}{4}-cfb)xc(\frac{N}{4}-cfb)xc}{fm \qquad fm}$

=10.5+$\frac{71.25-8471.25-84}{29 \qquad 29}$x5=14.6810millions

Q3= L+$\frac{(\frac{N}{2}-cfb)xc(\frac{N}{2}-cfb)xc}{fm \qquad fm}$

=25.5+$\frac{213.75-162213.75-162}{53 \qquad 53}$x5=30.3820 millions

Substitutions in the formula

SK = $\frac{Q3+Q1-2mdQ3+Q1-2md}{Q3-Q1 \qquad Q3-Q1}$

=$\frac{30.380+14.6810-2x23468830.380+14.6810-2x234688}{30.3820-14.6810 \qquad 30.3820-14.6810}$=-0.1194

Negative sign indicate negative skewed

Note : since mean is great than median skew is positive

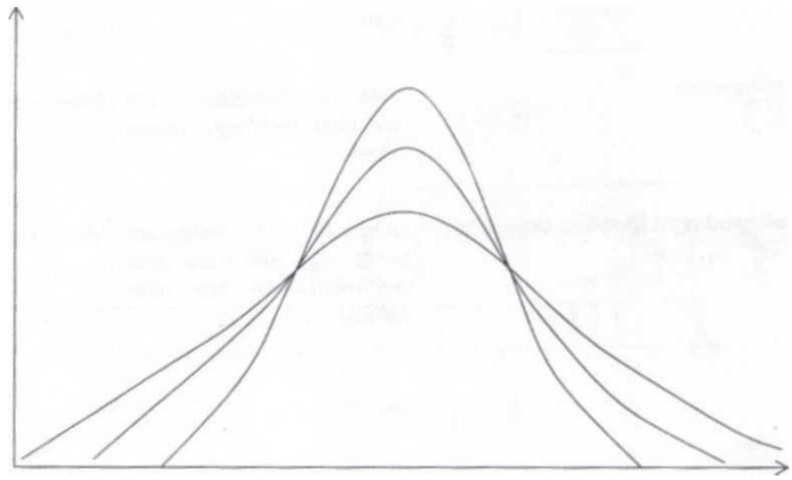
Comparison of distribution

Distribution's shape	Histogram Appearance	Statistics

Skewed left (negative skewness)	Long tail of histogram points left with a few low values but most data on right	Mean < median
Symmetric	Tails of histogram are balanced (low/high values offset)	Mean = median
Skewed right (positive skewness)	Long tail of histogram points right with most data on left but a few high values)	Mean > median

12.3.Kurtosis

Kurtosis refers to the peakedness of a distribution. Sometimes it may be necessary to distinguish measures of skewness and kurtosis from those corresponding to a population of which the sample was part. It becomes necessary to use their frequency curve. Kurtosis is the peakedness (quality of flatness) of a distribution relative to the mean. Frequency curves may have several different bell shapes. The most common is a bell shaped normal curve. The other curve which is characterised by piling at the centre of the distribution is referred to as leptokurtic Diagrammatically, it is more peaked than the normal curve. The other is gentler and flatter, always below the normal curve is called platykurtic. The curve that assume the shape of the normal curve is called mesokurtic. In general, the level of the peak on curve of the distribution determines its kurtosis.



A = Mesokurtic (normal peakedness)

B = Platykurtic (flatter than normal)

C = Leptokurtic (more peaked than normal)

Self test questions

Question 12.1

what is meant by the term skewed distribution?
Solution: The answer to this question is shown in Chapter 8, Section 2.
Question 12.2

What do each of the following population shapes look like?

a) Symmetrical and bell shaped.
b)Double-peaked,
c) Skewed with a tail to the left.
d)Skewed with a tail to the right.

Solution:
The answers to these questions lie in Chapter 8, Sections 2 and 5.

Question 12.3

A shop attendant took record of his daily sales and found that the sales have a mean of 20, a median of 22 and a standard deviation of 10. Required to determine the coefficient of skewness of the sales distribution.

Solution:

The answer is - 3, therefore, a negative value shows negative skewed.

Question 12.4

A data set for a distribution on distance covered by a vehicle has a mean of 454.5 km, a median of 452.4 km and standard deviation of 27 km. What is the direction and degree of skew of this distribution?

Solution:
The answer is 4, therefore, positive values show positive direction degree 0.0778

Question 12.5

Explain how the population mean, median, and mode compare when the population's relative frequency curve is:

a) symmetrical;
b) skewed with a tail to the left;
c) skewed with a tail to the right; and
d) Normally distributed.

Solution: The answers to these questions lie in Chapter 8, Section 5.

Question 12.6

Define the terms:

a) Kurtosis;
b) Mesokurtic;
c) Platykurtic; and
d) Leptokurtic.

Solution:
The answers to these questions can be found in Chapter 8, Section 6.

Question 12.7

In a distribution, the difference between two quartiles is 15 and their sum is 35 and the median is 20.

Required:
Find the coefficient of skewness and comment on the skewness of the distribution.

Solution:
The coefficient of skewness is - 0.3333. The negative sign shows negative skewness.

CHAP 12.

SET THEORY AND VENN DIAGRAMS

A Set is a collection of distinct items or objects e.g. members, letters, people, houses etc. The items or objects in a set are called members or elements of the set. Any set is denoted using a capital letter while the elements are denoted using small letters. The members or elements of the set are enclosed within the curly brackets and separated using comas, e.g. a set of vowels can be written as follows; A = {a, e, i, o, u}. If element x is a member of set A it is denoted as follows x ∈∈ A (x belongs to set A). If X is not an element of A it is denoted as x ∉x ∉A (x doesn't belong to set A).

We may consider all the ocean in the world to be a set with the objects being whales, sea plants, sharks, octopus etc, similarly all the fresh water lakes in Africa can form a set. Supposing A to be a set

A = {4, 6, 8, 13}

The objects in the set, that is, the integers 4, 6, 8 and 13 are referred to as the members or elements of the set. The elements of a set can be listed in any order. For example,

A = {4, 6, 8, 13} = {8, 4, 13, 6}

Sets are always precisely defined. Each element occurs once and only once in a set.

The notation ∈ is used to indicate membership of a set. ∈ represents non membership. However, in order to represent the fact that one set is a subject of another set, we use the notation ⊂. A set "S" is a subset of another set "T" if every element in "S" is a member of "T"

Example 12.1

If A = {4, 6, 8, 13} then

i) 4 ∈ {4, 6, 8, 13} or 4 ∈ A; 16 ∈ A

ii) {4, 8} ⊂ A; {5, 7} ⊄ A; A ⊂ A

Methods of set representation

Capital letters are normally used to represent sets. However, there are two different methods for representing members of a set:

i. The descriptive method and

ii. The enumerative method

The descriptive method involves the description of members of the set in such a way that one can determine the elements of the set without difficulty.

The enumerative method requires that one writes out all the members of the set within the curly brackets.

For example, the set of numbers 0, 1, 2, 3, 4, 5, 6 and 7 can be represented as follows

P = {0, 1, 2, 3, 4, 5, 6, 7} , enumerative method

P = {X/*x* = 0, 1, 2…7}   descriptive method

Or

P = {x/0 ≤ *x* ≤7} *where x is an integer.*

Application of set Theory

i) It is used in capturing statistical data.

ii) It is used in solving counting problems

iii) It shows the logical relationship between two or more sets.

iv) It creates a basis for probability theory

v) It is a research tool that can be used in data capturing.

12.1. Types of sets

Subset – This is a portion of a set where the elements of that set belong to another bigger set.

Universal set (U) – This is a set containing all the elements under consideration e.g. a set of all the students in college, a set of alphabetical letters, a set of all the months in the source of the year.

Finite set – This is a set containing countable elements e.g. a set of weekdays a set of students in sec iv etc.

Null/Empty /void set (∅∅) – A set without elements, e.g. a set of married bachelors.

Infinite sets – This is a set containing countless elements e.g. a set of counting numbers.

Sets concepts and Operations

Concepts:

Overlapping sets

These are two or more sets with some common elements.

Eg: A{1,2,3,4,5,6}

B{2,4,6,8,10}   Overlapping set.

Sets equality

Two or more sets are said to be equal if and only if they have the same elements but not necessarily the same order of elements.

Eg: A- {a, b, c, d}

C = {b,c, a, d,}

A = C

Disjoint sets

These are two or more sets without common elements

Eg: A- {a, b, c, d}

C = {1,2, 3, 4,}

Set operation;

Sets intersection (n)

This operation represents a set containing the common elements in two or more sets.

If A = {1 2 3 4 5 6}

B = {2, 4, 6, 8, 10}

Then AnB = {2 4 6}

If set C = {11, 12, 13,14}

Then AnC =(ØØ)

Set Union

This operation represents a collection of all the elements in two or more sets without repetition if the sets are overlapping.

If A = {1 2 3 4 5 6} ⟹⟹ n (A) = 6

B = { 2, 4, 6, 8, 10}⟹⟹n (B) = 5

AUB = {1, 2, 3, 4, 5, 6, 8, 10} ⟹⟹ n(AUB) = 8

Set difference (-)

Given two sets A & B which are overlapping, the difference between A & B is a set of elements that are in set A but not in set B.

Similarly B difference A is a set of elements in B but not in A.

If A = {1, 2, 3, 4, 5, 6}

B= {2, 4, 6, 8, 10}

Then A – B = {1, 3, 5}

B – A = {8, 10}

Compliment (C)

Compliment of a set is a set of elements that are not in the original set but they are part of the universal set, e.g.

If A = {1, 2, 3, 4, 5, 6}

Then compliment of A = A^c = A1 = {7, 8, 9, 10 ....∝∝ }

NB: Set theory begins with a fundamental binary relation between an object *o* and a set *A*. If *o* is a **member** (or **element**) of *A*, write *o*□*A*. Since sets are objects, the membership relation can relate sets as well.

A derived binary relation between two sets is the subset relation, also called **set inclusion**. If all the members of set *A* are also members of set *B*, then *A* is a **subset** of *B*, denoted *A*□*B*. For example, {1,2} is a subset of {1,2,3} , but {1,4} is not. From this definition, it is clear that a set is a subset of itself; for cases where one wishes to rule out this, the term **proper subset** is defined. *A* is called a **proper subset** of *B* if and only if *A* is a subset of *B*, but *B* is **not** a subset of *A*.

Just as arithmetic features binary operations on numbers, set theory features binary operations on sets. The:

- **Union** of the sets *A* and *B*, denoted *A*□*B*, is the set of all objects that are a member of *A*, or *B*, or both. The union of {1, 2, 3} and {2, 3, 4} is the set {1, 2, 3, 4} .

- **Intersection** of the sets *A* and *B*, denoted *A* ∩ *B*, is the set of all objects that are members of both *A* and *B*. The intersection of {1, 2, 3} and {2, 3, 4} is the set {2, 3} .

- **Set difference** of *U* and *A*, denoted *U* \ *A*, is the set of all members of *U* that are not members of *A*. The set difference {1,2,3} \ {2,3,4} is {1} , while, conversely, the set difference {2,3,4} \ {1,2,3} is {4} . When *A* is a subset of *U*, the set difference *U* \ *A* is also called the **complement** of *A* in *U*. In this case, if the choice of *U* is clear from the context, the notation A^c is sometimes used instead of *U* \ *A*, particularly if *U* is a universal set as in the study of Venn diagrams.

- **Symmetric difference** of sets *A* and *B*, denoted *A*□*B* or *A*□*B*, is the set of all objects that are a member of exactly one of *A* and *B* (elements which are in one of the sets, but not in both). For instance, for the sets {1,2,3} and {2,3,4} , the symmetric difference set is {1,4} . It is the set difference of the union and the intersection, (*A*□*B*) \ (*A* ∩ *B*) or (*A* \ *B*) □ (*B* \ *A*).

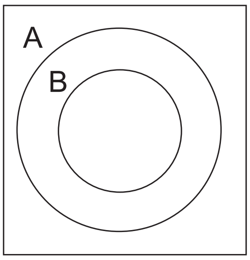
- **Cartesian product** of *A* and *B*, denoted *A* × *B*, is the set whose members are all possible ordered pairs (*a*,*b*) where *a* is a member of *A* and *b* is a member of *B*. The cartesian product of {1, 2} and {red, white} is {(1, red), (1, white), (2, red), (2, white)}.

- **Power set** of a set *A* is the set whose members are all possible subsets of *A*. For example, the power set of {1, 2} is { {}, {1}, {2}, {1,2} } .

Some basic sets of central importance are the empty set (the unique set containing no elements), the set of natural numbers, and the set of real numbers.

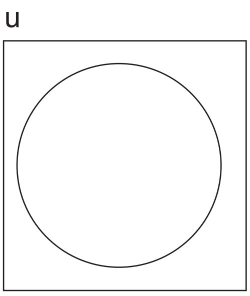
12.1.1 Venn diagrams

This is a pictorial representation of sets and their relationships. They involve the use of loops enclosed within a square or a rectangle. The loop represent a specific set while the square / rectangle represents the universal set from where the set was drawn.



If set B is a subset of A then the venn diagram of subset B is (BCA).

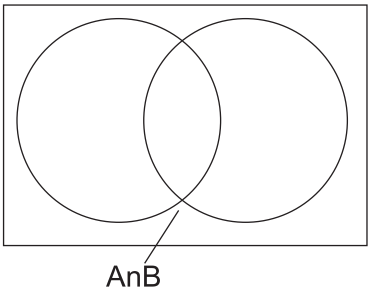
Set A

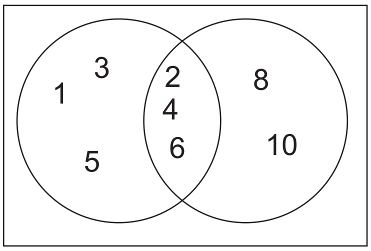


Intersection of set A & B (AnB) (overlapping sets)

IF A = {1, 2, 3, 4 ,5, 6}

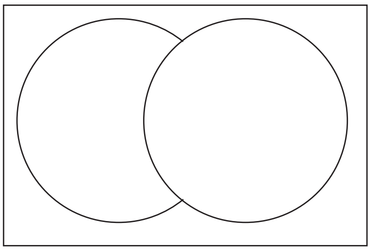
B= {2, 4, 6, 8, 10}



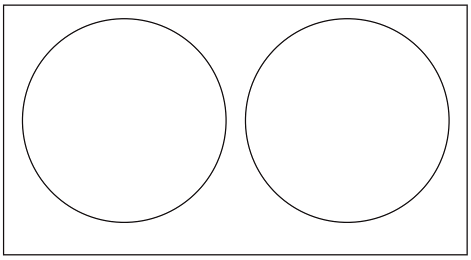


Then;

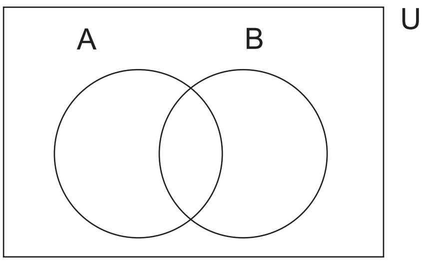
AUB (A union B) (Overlapping sets)



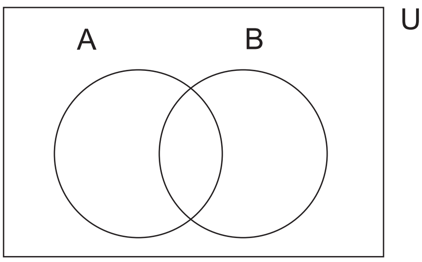
AUB (Disjoint Sets)



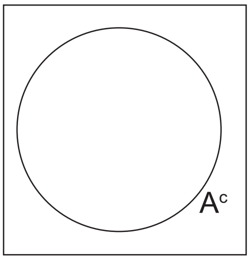
A – B (over lapping sets) i.e A difference B



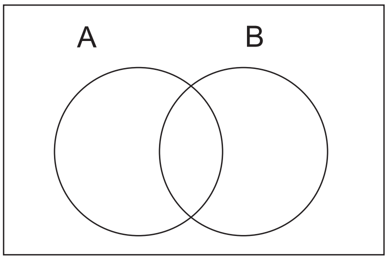
B – A



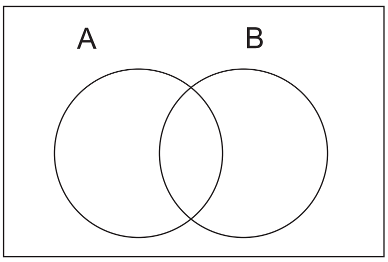
Complement of A (A^c)



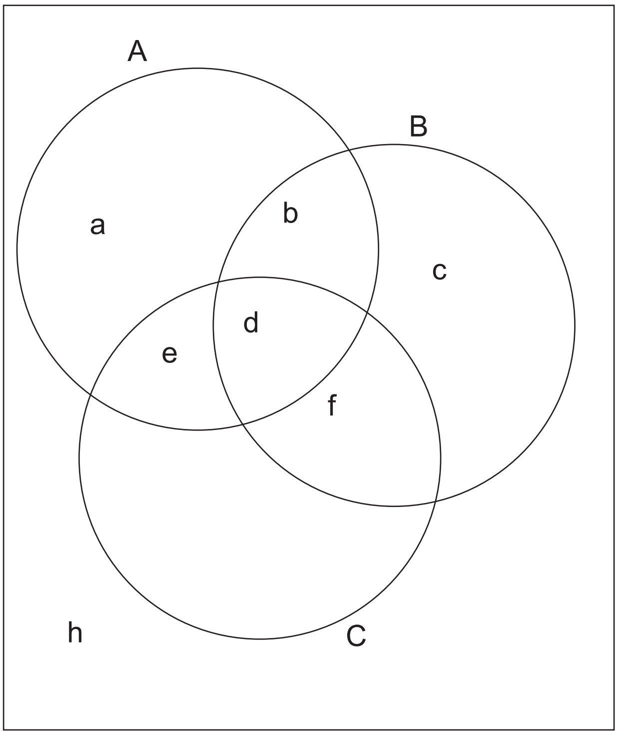
$(A \cup B)^C$



$(A \cap B)^C$



Venn diagram for sets A, B, & C (overlapping set)



Observation

The venn diagram has 8 sectors i.e: a, b, c, d, e, f, g, & h. The small letters represents number of elements in each sector.

Sector	Interpretation
a, c and g	– Number of elements in set A only, B only and C only.
b, e and f	– Number of elements at the intersection of A and B only, A and C only, B and C only respectively. eb = AnB-C; e = AnC-B; f = BnC – A.
d	– Number of common elements in all the three sets i.e. AnBnC
h	– Number of elements outside the three sets i.e. (AUBUC)C
	–
b+d	– AnB (A and B)
d + e	– AnC (A and C)
d + f	– BnC
a + b + c	– A or B only. (AuB only) (AUB – C) Same as c + f + g and a + e + g
a + b + d + e	– A
c+ b + d + f	– B

	–
a + b+c +d+e +f +g + h	– U (Universal set)
a+b+c + d + e + f	– A or B (AUB)

12.2. Solving problems using venn diagrams

Illustration

a) A quick survey of 1,000 children in a refugee camp produced the following results:
- 320 children were fed on beans
- 200 children were fed on rice.
- 450 children were fed on potatoes.
- 150 children were fed on beans and potatoes.
- 70 children were fed on beans and rice.
- 100 children were fed on rice and potatoes.
- 300 children were fed on none of the three types of food.

Required:

(i) Present the above information in the form of a Venn diagram.

(ii) The number of children who were fed on all the three types of food.

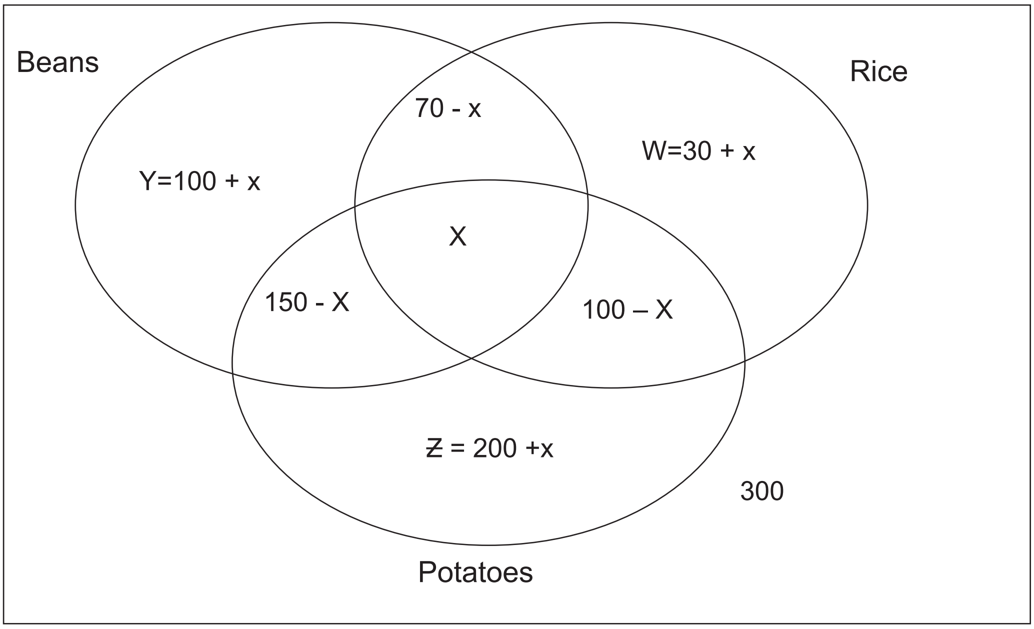
(iii) The number of children who were fed on exactly one of the three types of food.

(iv) The number of children who were fed on at least two types of food.

Solution

VENN DIAGRAM

$\epsilon = 1000$



Y + 70 – X + X + 150 – X = 320

Y – X = 320 – 220

Y = 100 + X

W + 70 – X + X + 100 – X = 200

W = 30 + X

W = 200 – 170 + X

Z + 150 – X + X + 100 – X = 450

Z = 450 – 250+X

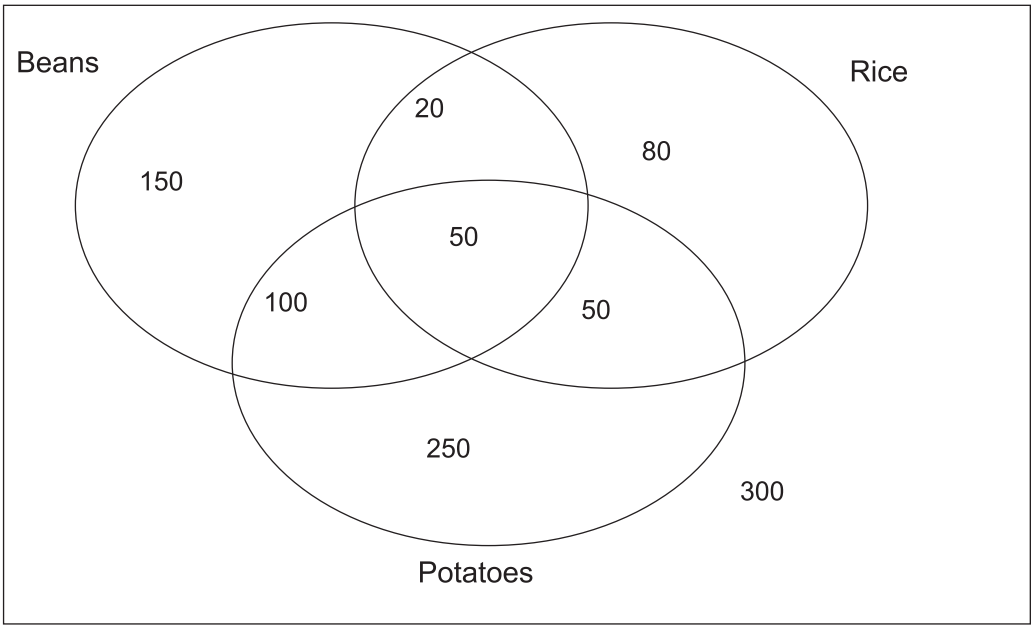
Z= 200 + X

100 + X + 70 – X + X + 150 – X + 30 + X + 200 + X + 300 = 1,000

X = 1,000 – 950

X = 50

CORRECT VENN DIAGRAM



ii) The number of children who fed on all the three types of food = 50

i) The number of children who fed on exactly one of the three types of food

150 +50 + 250 = 480

ii) The number of children who fed on at least two types of food 100 + 20 + 50 + 50 = 220

CHAP 13

PROBABILITY THEORY

13.1. Study objectives

By the end of this chapter, you should be able to:

- describe the basic concept of probability including the classical definition;
- identify properties of probability theory: probability limits, total probability and complementary;
- define the terms event, outcome, sample, sample space and equiprobable event as used in probability theory;
- determine the probability that an event will occur;
- apply the definition and rules of probability;
- determine types of events;
- use the tree diagrams in the calculations of probabilities; and
- Calculate and apply expectation of a given distribution.

Uncertainty plays an important role in our daily lives and activities as well as in business. In business context, investors cannot be sure which of the stocks will yield the best over the coming year. The engineers will try to reduce the likelihood that a machine will break down. Likewise, marketers may be uncertain as to the effectiveness of an advertisement campaign or the eventual success of a new product. Probabilities have proven vital in many ways to both business and the social sciences based on uncertainties.

13.2. Basic concepts of probability

a) Random experiment

Random experiment is any well-defined procedure that produces an observable outcome that could not be perfectly predicted in advance. A random experiment must be well defined. It must produce a definite, observable outcome so that one knows what happened after the random experiment is conducted.

For example, a test administered to a group of students in order to find out their academic strength. The outcome that could not be predicted before is whether the class is very strong, strong, average or weak. The outcome of the test administered will bring out the possible results.

b) Outcome

An outcome is the result of the random experiment describing and summarizing the observable consequences. Each time a random experiment is conducted, it produces exactly one outcome. For instance:

- in tossing a coin, the possible outcome would be either a head or a tail;
- the outcome of a march played by a team would be either a win, draw or a loss; and
- the results at an interview sat would be either a success or a failure

c) Sample space

Sample space is a list of all possible outcomes of the random experiment.

For example, a person doing business may get profits, losses or neither losses nor profits. The three possibilities form a sample space.

d) Event

An event is a set of outcomes to which a probability is assigned. An event is denoted by letter E.

For instance, when a die is tossed the probability of getting a three is 1/6. In this case the event is getting a three on the die.

e) **Null event**

This is an event whose probability is zero. For example, finding a worker aged 180 years in an organization.

f) **Definite event**

This is an event whose probability is one. For example, the sun rising from the east.

g) **Equiprobable events**

These are events that are equally likely to occur (they have equal probabilities). For example, when an unbiased coin is tossed once the probability of a head occurring = the probability of getting a tail = 1/2.

13.3. Classical definition of probability

The probability P of an occurrence or event E is the number of favorable outcomes divided by the number of possible outcomes.
Symbolically written as:

$$P(E) = \frac{\text{Number of favorable outcome}}{\text{total number of possible outocom}} \frac{n(E}{n(S} \frac{\text{Number of favorable outcome}}{\text{total number of possible outocom}} \frac{n(E}{n(S}$$

13.4. Properties of probability

a) Probability limits

The probability of an event must lie between 0 to 1.

That is $0 \leq P(E) \leq \leq P(E) \leq 1$

A probability close to 0 indicates that the event is not likely to occur. A probability close to one indicates that the event is most likely to happen.

b) **Complementary events**

The complement of an event A is the event that A does not occur. This is denoted as A^1 or A^c. If the probability of A is P(A) then the $P(A^1) = 1 - P(A)$.

Example 13.1

The probability of a product being of acceptable quality is 2/3, therefore, the probability of a product not being of acceptable quality= 1 - 2/3 = 1/3.

c) Exhaustive events

These are events whose sum of the probabilities of all possible outcomes in a sample space is one.

Example 13. 2

When a die is thrown once, the possible outcomes are 1, 2, 3, 4, 5, 6.
P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = $\frac{11}{[illegible]}+\frac{11}{[illegible]}+\frac{11}{[illegible]}+\frac{11}{[illegible]}+\frac{11}{[illegible]}+\frac{11}{[illegible]}=1$

Example 13.3
The managing director of an organization is about to retire. The probability that the position will be replaced by a female, P (F) = 0.47 or by male, P (M)=0.53. Find P (F or M).
P(F or M) = P(F) + P(M) = 0.47 + 0.53 = 1.00

d) Independent events
These are events where the occurrence of one does not affect or influence the occurrence of the others. For example, the performance of female students and the male students in an examination.

For two independent events A and B the probability that both events occur is written as P(A and B) or P(A∩B).
By the multiplication rule P(A∩B) = P(A)xP(B)

Example 13.4

If two events A and B are independent such that P(A)=0.35 and P(B)=0.6,find the probability that both events occur.
P(A∩B) = P(A)xP(B)
= 0.35 x0.6 = 0.21

Example 13.5

A coin is tossed and a six-sided die is rolled. Find the probability of getting a head on a coin and a six on the die.

Solution:
Let H be the event of getting a head on a coin; and Let six be the event of getting a six on a die.

Note: These two events are independent because the occurrence of one does not affect the occurrence of the other.

P(H) = 1/2

P(6) =1/6

P(H and 6) = $\frac{1}{2}x\frac{1}{6} = 1/12\frac{1}{2}x\frac{1}{6} = 1/12$

Example 13.6

A bag contains five blue (B) marbles and ten red **(R)** marbles. A marble is chosen at random from the bag. After replacing it, a second marble is chosen. Find the probability of choosing:

i) both red; and
ii) first is red and second is blue.

Solution:
i) P(both are red) = P**(R1**and **R2) (R1** and **R2** are independent events)
= P**(R$_1$**)xP(R$_2$) **=10/15x10/15**
(selection with replacement does not affect the sample space of the marbles in marbles in the bag)

=4/9 (simplified form)

ii) P (First red and second is blue) = P(R1 and B2) (R1 and B2 are I independent events)
= P(R,) XP (B$_2$)
=1/15x10/15 (selection with replacement does not affect the sample space of the marbles in the bag)
=2/9 (simplified form) e)

c) Mutually exclusive events

These are events that cannot occur at the same time.

Example 13.7
A corporate ending a year with either “very high” or “very low” profits.

Example 13.8
A student “passing” or “failing” same exam

For two mutually exclusive events A and B, the probability that either A or B will occur is written as P(AUB). This is obtained using the addition rule
P(AUB) =P(A) + P(B)

Example 13.9

A room has three red, two blue and four white plastic chairs. A person selects a chair at random.

Required:
Find the probability that a chair selected will be either blue or white
Solution:
Let R. B and W be the events of selecting a red or blue or white chair respectively

Then P(R) =$\frac{33}{99}$P(B) =$\frac{22}{99}$and P(W) =$\frac{44}{99}$

P (B or w) =$\frac{22}{99}$+$\frac{44}{99}$=$\frac{66}{99}$= $\frac{22}{33}$

d) Dependent events

These are events where the occurrence of one affects the occurrence of the other.

Examples:

- Being a smoker and getting cancer are dependent events because smokers are more likely to get cancer than non-smokers.
- The output of organization is dependent on the quality of its workers.
- A high salary pay is a known motivator for performance in any organisation.

13.5. Conditional probability

The conditional probability of an event A in relation to another event B is defined as the probability that event A occurs given that event B has already occurred.

For events A and B with respective probabilities of occurrence P (A), P(B) such that P(A)* 0 and P(B) 10. The conditional probability of A given that B has already occurred is written as P($A/_B$) is given by:

P(A/B) =$\frac{P(Anb)P(Anb)}{P(B) \quad P(B)}$

Note:

i)When events A and B are independent, P(A$\cap$ B) = P(A)x P(B), hence P(A/B) = P(A)

ii)When the events A and B are mutually exclusive, P(A^n B) =0, hence P($A/_B$) = 0

Example 13.10

A die is tossed once.

Required:

Find the probability that the number that shows up is a multiple of three given that it is an even number.

Solution:

Let E be the event that an even number shows up, M be the event that the number is a multiple of three, then

$S_=$ {1,2,3,4.5,6} $E_=$ {2,4,6}$M_=$ {3,6}

P(E) 3/6, pm =2/6, P(E^{n}M)=1/6

P(M/E) =$\frac{P(EnM)P(EnM)}{P(M) \quad P(M)}$

Example 13.11

In a certain village, the probability that a person is a registered voter and is a member of a village council is 0.042. If the probability that a person is a member of a village council is 0.21.

Required: If a person is a member of the village council find the probability that a person is a registered voter.

Solution:

Given that P(C^V) = 0.042, P(C) = 0.21;

$$P(V/C) = \frac{P(C \cap V)}{P(c)}$$

Then

$\frac{0.042 \, 0.042}{0.21 \quad 0.21}$=0.2

13.6. Tree diagrams

A tree diagram is a figure that branches from a single root. It is used to list all possibilities of a sequence of events in a systematic way. It is a convenient and a useful approach to a number of probability problems.

Example 13.12

A box contains five white and four yellow identical bean seeds. Two seeds are drawn, one after the other.

Required:

Find the probability of drawing one white and **one** yellow seed, if the first is:

a) replaced; and

b) not replaced

Solution:

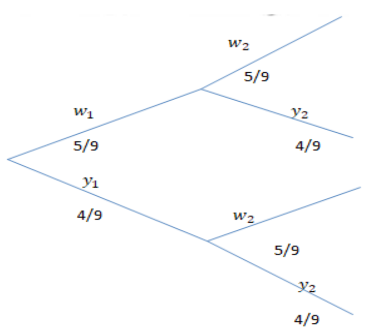
Let **Wi** be the event the **white** seed is drawn first

W2 be the event the white seed is drawn second

Y1 be the event the yellow seed is drawn first

Y2 be the event the yellow seed is drawn second

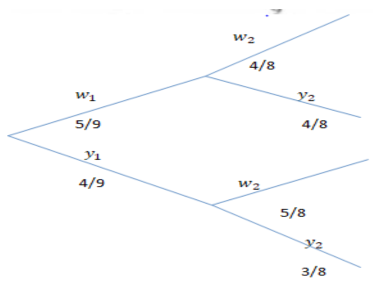
a) Tree diagram (picking with replacement



P(one white and one yellow seed) = $\boldsymbol{P}(W1 \cap Y2) + \boldsymbol{P}(Y1 \cap W2)$.= $\boldsymbol{P}(W1 \cap Y2) + \boldsymbol{P}(Y1 \cap W2)$.

$= \frac{55}{99} X \frac{44}{99} + \frac{44}{99} X \frac{55}{99} = \frac{4040}{8181}$

b) Tree diagram 1(picking without replacement)



P(one white and one yellow seed)

= $P(W1 \cap Y2) + P(Y1 \cap W2). P(W1 \cap Y2) + P(Y1 \cap W2).$

5/9x4/8+4/9x5/8=5/9

Example 13.13

A coin is tossed three times.

Required:

a) Draw a tree diagram and list the sample space.

b) Hence, find the probability of obtaining:

- exactly two tails; and
- at least two heads.

Solution:

Definition of events:

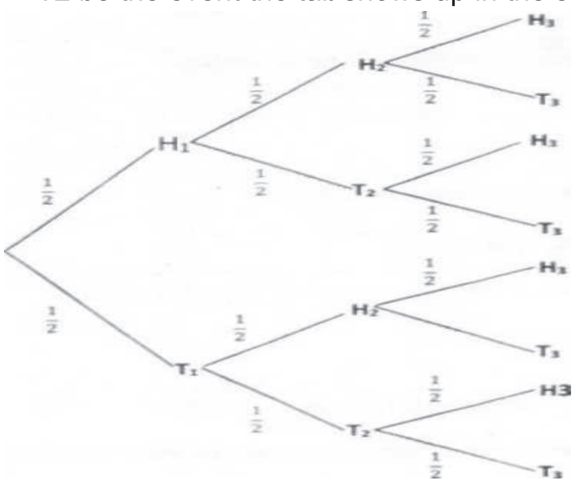
Let Hi be the event the head shows up in the first toss

H2 be the event the head shows up in the second toss

H3 be the event the head shows up in the third toss

T1be the event the tail shows up in the first toss

T2 be the event the tail shows up in the second toss



T3 be the event the tail shows up in the third tossx

The tree diagram 2:

The example

S = (H1H2H3, H1H2T3, H1T2H3, H1T2T3, T1H2H3, T1H2T3, TiT_2H_{3l}T1T2T3).

Required:

If the job delayed beyond the scheduled date of completion, what is the probability that it is organization B that caused it?

Solution:

Let A be event organization A did the job
B be event organization B did the job
C be event organization C did the job
D be event the job delayed beyond the scheduled date of completion

Then P(A) = 0.60 P(D/A) = 0.50
P(B) = 0.25 P(D/B) = 0.30
P(C) = 0.15 P(D/C) = 0.10

$$P(B/D)\ \frac{P\left(\frac{D}{B}\right) X\, P(B)\, P\left(\frac{D}{B}\right) X\, P(B)}{P(D) \qquad P(D)}$$

But P(D) = P(D/A) X P(A) + P(D/B) X P(B) + P(D/C) X P(C) = (0.50)(0.60) + (0.30)(0.25) + (0.10)(0.15) = 0.39

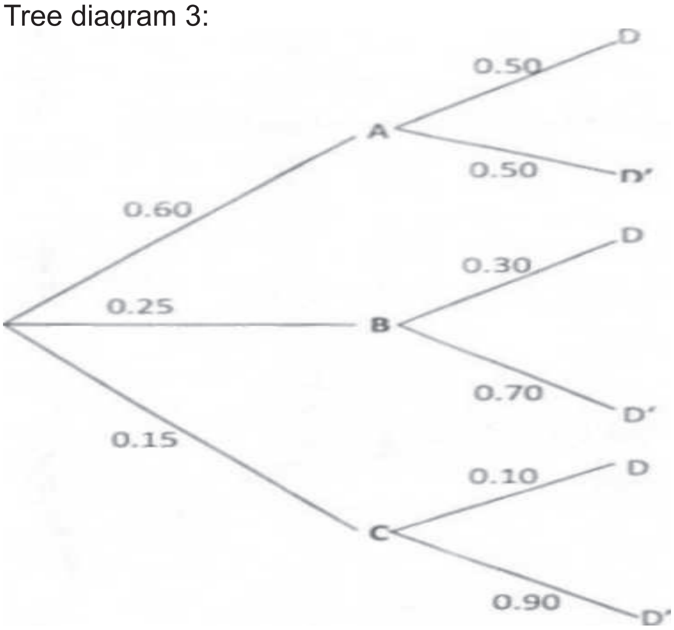
Then P(B/D) = 0.075/0.39

5/26(or 0.1923)

Three business organisations A,B and Care hired to do a job for company yy Okot Ltd. They are able to handle 60%,25% and 15% of the Job, respectively. The Probabilities that A,B and C caused the job to delay beyond the scheduled date of completion is 0.5,0.3 and 0.1 respectively

Alternatively, a tree diagram may be used to answer the above problem as illustrated below.

Tree diagram 3:



from p (B/D) = $\frac{P(B\cap D)P(B\cap D)}{P(D) \quad P(D)}$

$$= \frac{P(B\cap D) \qquad\qquad P(B\cap D)}{P(A\cap D)+P(B\cap D)+P(C\cap D)P(A\cap D)+P(B\cap D)+P(C\cap D)}$$

$$= \frac{(O.5O)(0.30)}{(0.60)(0.50)+(0.25)(0.30) + (0.15)(o.10)} =$$

Hence, P(B/D) = **0.1923.**

13.7. Prior and posterior probabilities

Consider an experiment, whose sample space has n mutually exclusive partitions **Ai,** A2, A3**….,A_n,** with associated probabilities P**(A_1),** P(A_2), P(A_3),…..,P**(An).** These probabilities are called prior probabilities because they are determined before any new information is taken into account. From tree diagram **3,** P(A) = **0.60** is prior probability.

However, the probability that has been revised based on new information is referred to as posterior probability**.** This is because it represents a probability calculated after new information is taken into account Also in tree diagram **.3** the P (D/A) = 0.60 **x** 0.50 = 0.30is posterior probability.

Therefore, using Bayes' theorem a prior (unconditional) probability that an event Ai will occur or has occurred is revised to a new probability, posterior (conditional) probability, for example, P(A1/E), which is the probability that event Ai occurs given that event has already occurred

13.8. Self-test questions

Question 13.1

Define the following terms as applied in probability theory:

;) Null event.

ii) Definite event.

iii) Mutually exclusive events.

Solution:

The answers to i) and ii) can be found in Chapter 9. Section 1d) and the answer to iii) can be •'ound in Chapter 9, Section 3e).

Question 13.2

A certain professor taught quantitative techniques and business management courses. The z'ade distribution per course is as shown below:

	Course	
Grade	**Quantitative techniques**	**Business management**
A	7	8
B	9	10
C	11	12
D	6	9
0	5	8

A student who was in the professor's class during the semester is randomly selected.

Required:

Find the probability that the student:

i) received grade A;

ii) was in the Quantitative Techniques class; and

iii) was in the Quantitative Techniques class and received grade A.

Solution:

i) The probability of the student receiving a grade A is $\frac{3\ 3}{1717}$

ii) The probability of the student being in the Quantitative Techniques class is $\frac{3\ 3}{1717}$.

iii) The probability of the student being in the Quantitative Techniques class and receiving **a grade** $\frac{7\ 7}{8585}$

Question 13.3

In order to establish the competitiveness of 3. (a) In order to establish the reading habits of CPA(R) students, a sample of 50 students was selected and asked to name the newspapers they read regularly. The results obtained showed that 25 read The New Vision (N), 16 read PThe Daily Monitor (M), 14 read The Observer (O); 5 read both N and M, 4 read both M and O, -and 6 read both N and O; and 2 read all

the three.

Required:

Find the probability that a student selected in this sample reads:

i) at least one of the newspapers,

ii) only one of the newspapers

iii) Only The New Vision.

Solution:

i) The probability that a student reads at least one of the newspapers is $\frac{2121}{2525}$.

ii) The probability that a student reads only one of the newspapers is $\frac{3131}{5050}$

iii) The probability that a student reads only The New Vision is $\frac{3131}{5050}$

Question 13.4

Yellow Telecom, Purple Telecom and Red Telecom mobile telephone companies in the Rwanda n market, a survey was conducted on a sample of customers selected from Kigali. The results generated were as follows: One hundred and ninety were Yellow Telecom customers, 205 were for Purple Telecom and 260 for Red Telecom. Fifteen customerswere for all the three networks, 55 were for Yellow Telecom and Red Telecom, 100 for Yellow Telecom only, 135 were for Red Telecom only. One hundred and eighty of the people under study were not on any network.

Required:

Determine the:

i) size of the sample.

ii) probability that a customer is on Purple Telecom only,

iii) probability that a customer is on one network only,

iv) probability that a customer is on two networks only

Solution:

i) the size of the sample is 660.

ii) The probability that a customer is on Purple Telecom only is $\frac{17\ 17}{132132}$

iii) The probability that a customer is on one network only is $\frac{1616}{3333}$

iv) The probability that a customer is on two networks only is $\frac{29\ 29}{132132}$

Question 13.5

After completing the CPA(R) course, a student applied for a job in two different organizations. The probability that he/she gets a job in the first organization is and the probability that he/she gets a job in the second organization is $\frac{44}{55}$

Required:

Find the probability that he/she gets:

i) both jobs

ii) only one job

Solution:

i) The probability that he/she gets both jobs is $\frac{8\ 8}{2525}$

ii) The probability that he/she gets only one job is $\frac{1414}{2525}$

. **Question 13.6**

Given the events A and B such that P (A/B) = $\frac{22}{55}$, P (B) =$\frac{11}{44}$ and P(A) =$\frac{11}{33}$,

Required:

Find:

i) P(A∩B)

ii) P(AUB)

iii) P(B/A)

Solution

❖ $\frac{1\ 1}{1010}$

❖ $\frac{2929}{6060}$

❖ $\frac{3\ 3}{1010}$

Question 13.7

Dauglas and Aristock buy used laptops for company X. On a certain day Dauglas bought 60% for the laptops of which 15% had faults while 25% of those bought by Aristock had faults too. A laptop was selected at random.

Required:

Find the probability that the laptop:

a) had a fault,

b) That had fault was bought by Dauglas.

Solution:

a) 0.1900.
b) 0.4737.

Question 13.8

Three auditors Marvin, Joshua and Faustine share an office table. During their first meeting in a year the senior auditor, who is their supervisor among other items pens as shown in the table below:

	Pens	
Auditor	Red	Green
Marvin	80	20
Joshua	30	40
Faustine	10	60

The supervisor realises that she has given out all the pens and yet she needed one to use immediately. She decides first to pick an auditor at random and then randomly pick a pen.

Required:
a) Find the probability that the pen picked is red.
b) If she picks a red pen, what is the probability that it is Joshua's?
Solution:

a) 0.4571.
b) 0.3125.

Question *13.*9

i) Given **Ai,** *A2,* and A3 are mutually exclusive and exhaustive events of a sample space **S.** If E is any event associated with S, such that P(E) **#0,** state Baye's theorem.

ii) Roberto can travel to work by Taxi or Boda or on foot with respective probabilities $\frac{11}{22}$, He arrives late when uses any of these means of transport with respective probabilities $\frac{33}{55}$, $\frac{3\ 3}{1010}$ and $\frac{1\ 1}{1010}$. If he arrives late on a certain day, what is the probability that hetravelled by foot?

Solution:

i)P(A1/E)= $\frac{P(A1)P(\frac{[illegible]}{A1})}{\sum B [illegible]}=\frac{P(A1)P(\frac{[illegible]}{A1})}{\sum B [illegible]}$

ii) $\frac{2525}{[illegible]}$

Question 13.10

A bag contains 15 apples, of which 3 are brown and 12 are green. *An* apple *is* picked at random from the bag and its colour recorded. The apple is not replaced. *A* second apple *is* then picked out at random, find the probability that:

a) the first apple *is* brown; and
b) the second apple is green given that the first apple was brown.

PROBABILITY DISTRIBUTION (RANDOM VARIABLES)

14.1. Study objectives

By the end of this chapter, you should be able to:

Distinguish between variables in probability distributions;

- Compute probabilities of random variables;
- Calculate expectation and standard deviation of random variables; and
- Determine expected gain or profit.

14.2. Basic concepts in random variables

It is important that before discussing probability distribution, the underlying concepts should be reviewed:

a) **Variable**

This is a characteristic or attribute that can assume different values. Letters of the alphabet such **as** X, Y, etc. can be used **to** represent variables. Variables associated with probability are called random variables.

b) **Random variable**

This is a variable whose values are determined by chance.
There are two types of random variables:
Discrete random variables and continuous random variables.

c) **Discrete random variable**

This is that variable that takes on countable domain such as the results from tossing a die. that is, 1, 2, 3, etc.

d) **Continuous random variable**

This is a variable that can have a value within a range. Such variables arise out of measurements of height, weight, length, time, etc.

The **probability distribution** of a discrete random variable X is a list of distinct numerical values of X along with their associated probabilities. Sometimes a formula can be used in place of a list.

Properties of a **probability distribution of a discrete random variable**

i) The probability that X takes on a particular value x is written as P(X=x) = P(X)

ii) The sum of the probabilities P(X) of all possible values of X must be equal to 1,

iii)

that is, $\sum p(x) = 1 \sum p(x) = 1$

iv) The probability P(X) must be between 0 and 1 i.e. 0 < P(X) < 1.

Example 14.1

Construct a probability distribution for tossing a single die.

Solution:

Since the sample space is 1, 2, 3, 4, 5, 6 and each outcome has a probability of $\frac{11}{66}$ then this distribution would be as shown:

Outcome X	1	2	3	4	5	6
Probability P(X)	1/6	1/6	1/6	1/6	1/6	1/6

A discrete probability function consists of the values a random variable can assume and its corresponding probability values. The probabilities can be determined by observation (experimentally) or theoretically.

Example 14.2

Given that {1,2,4}is a set of integers picked at random. Let X denote the sum of two of the integers from the given set.

i) List all distinct values of X.

ii) Obtain the probability distribution of X.

Solution:

i) Construct a table to show the sum of two integers:

+	1	2	4
1	**2**	**3**	**5**
2	**3**	**4**	**6**
4	**5**	**6**	**8**

iii) The possible outcomes of X are{2,3,4,5,6,8}.

X	**2**	**3**	**4**	**5**	**6**	**8**
P(X)	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

Example 14.3

The probability distribution of x is given by the formula

$P(x) = \frac{1}{15}\binom{4}{x}\frac{1}{15}\binom{4}{x}$ For x= 1,2,3,4

Determine the probability distribution.

Solution:

X	1	2	3	4
P(X)	4/15	6 /15	4/15	1/15

Exam

Example 14.4

The probability distribution of university students who read newspapers is as shown in the table below.

X	*0*	*1*	*2*	*3*	*4*
P(X)	***0.02***	***0.23***	***0.40***	***0.25***	***0.10***

Required:

Determine the:

i) P(X=2)

ii) P(X$\geq 2)\geq 2)$

iii)P(X$\leq 2)\leq 2)$

Solution :

i)P(X=2)=0.40

ii) P(X$\geq 2 = P(X = 2) + P(X = 3)\geq 2 = P(X = 2) + P(X = 3)$+ P(X=4)

=040+0.25+0.10=0.75

iii)P(X$\leq 2)\leq 2)$= P(X=2)+ P(X=1)+ P(X=2

=0.02+0.23+0.40

=0.65

14.3. Expectation of discrete random variable E(x)

Expectation is the mean of a given set of value .It is calculated using probabilities. The expectation of a discrete random variable X denoted E(X) is calculated using the formula E(x) =$\sum xP(X = X)\sum xP(X = X)$
The formula may also be written as e(x) =$\sum xiPi = 1,2 \ldots\ldots n\sum xiPi = 1,2 \ldots\ldots n$

Example 14.5

The table below shows the number of new tax payers registering with Rwanda Revenue Authority on a weekly basis.

No. of new tax payers per week(x)	0	1	2	**3**	**4**	**Total**
Probability(p)	0.20	0.36	0.30	0.12	0.02	**1.0**

Required:

Determine expected number **of tax** payers registering weekly.

Solution:

No. of new tax payers per week(x)	0	1	2	**3**	**4**	**Total**
Probability(p)	0.20	0.36	0.30	0.12	0.02	**1.0**
XP(X=X)	0	0.36	0.60	0.36	0.08	**1.4**

The sum$\sum$**xp(X=x)** = 1.4, is the expectation of the tax payers registered per week.

Example 14.6

Given the probability distribution below:

Find the expectation of X. **Solution:**

E(x) = £x P(X=x)

= -2(0.3) + (-1X0.1)+ 0(0.15) =1(0.4) + 2(0.05)

Hence, E(X) = -0.2

a) Properties of expectation

- E (a constant) = a constant, for example, E(9) = 9
- E(aX) = aE(X), for example, E(8X) = 8E(X)
- E(X + a) = E(X) + a, for example, E(X -7) = E(X) – 7

Example 14.7

From the following probability distribution

X	**1**	**3**	**5**	**7**	**9**
P(X=x)	**0.03**	**0.15**	**0.26**	**0.39**	**0.17**

Find the expected value of :

- **X**
- **x -1.5**
- **$^1/2X^2$**

Solution:

X	*P(X=x)*	*xP(X=x)*
1	0.03	0.03
3	0.15	0.45
5	0.26	1.30
7	0.39	2.73
9	0.17	1.54
	∑P(X=x) =***1.0***	∑xP(X=x) =***6.04***

i)E(X) = ∑xP(X=x) = 6.04

iv) E ($^{1/2}$ x^2) = $^{1/2\Sigma}x^2$P(X=x)

iii)E(X-1.5) = E(x)-1.5=6.04-1=5.99

X	1	3	5	7	9	
P(X=x)	0.03	0.15	0.26	0.39	0.17	
X^2P(X=x)	0.03	1.35	6.50	19.11	15.47	∑x^2P(X=x)=42.46

Then E (½ X^2) =1/2(42.46)

=21.23

Example 14.8

Aware house employee has weekend over time schedule of work. The probability that he will be given 6 hours of work is 1/6, 9 hours of work is 1/3or 12 hours of work is J4. Find the expected number of hours the employee will work over time in one weekend.

Solution:

Let X be the number of hours the employee works over time

X	6	9	12
P(X=x)	1/6	$1/_3$	1/2
xP(X=x)	1	3	6

Expected number of hours, E(x)=∑xP(X=x)
Thus, from the table above, E(X) = 1 +3 + 6 = 10 hours

Example 14.9

A milk vendor categorized his days according to financial gain as 'high', moderate' and 'low'. His estimates are such that the high days are 20% and moderate are 50%. He also calculated that his average revenue on the three categories of days were Frw 220,000, Frw 130,000 and Frw 40.000 respectively. If his average cost per day is Frw 80,000;

Required:
Calculate his expected profit per day.

Solution:
Let X be profit made by the vender per day.
Profit, x = average revenue - average cost:

	High	Moderate	Low
Profit, x	140,000	50,000	-40,000
Probability, P(X=x)	0.2	0.5	0.3
xP(X=x)	28,000	25,000	-12,000

Therefore, expected profit per day = Frw (28.000 + 25,000 - 12,000)
= Frw 41, 000

Example 14.10

On a certain promotional period, a mobile phone dealer expects the sale of their favorite hand set to follow the pattern:

Sale	0	100	200	300
Probability	0.1	0.4	0.3	0.2

The cost of the favorite hand set is $30 and was selling at $60 on promotion. Any remaining handsets would be sold at $5 each to clear off the handsets on promotion. If 200 were put up for promotion, calculate the dealer's expected profit.

Solution:
Let X be the profit made at any sales level (0, 100, 200, 300).
Now, profit on each hand set sold at $60 = $60 - $30 =$30,
Loss on each hand set sold at $ 5 = $30 - $5 = $25.
Profit made, at:
0 sales level = -$200(25) = -$5000
100 sales level = ${100(30) - 100(25)} = $500
200 sales level = $200(30) = $6000
300 sales level = $200(30) =$6000

Sales	0	100	200	300
X	-5000	500	6000	6000
P(X=x)	0.1	0.4	0.3	0.2

xP(X=x)	-500	200	1800	1200

Therefore, the dealer's expected profit:

$$E(X) = \sum xP(X = x)$$

= $(-500 + 200 +1800 + 1200)

= $2,700

14.4. Self –test questions

Question 14,1

x	0	1	2	3	4
P(X)	$\frac{1}{5}$	3a	$\frac{3}{10}$	a	$\frac{1}{5}$

P r o b a b i l i t y

The table above shows the distribution of a random variation X Find

i) The value of a and
ii) expectation of X

Solution:

i) $\frac{3}{25}$

ii) $\frac{77}{55}$

Question 14.2

Give that the expectation of a random variable x E(X) =4 find
i)E(2X)
ii)E(4X+2)

Solution:

i) 8
ii) 18

Question 14.3

A whole sale shop on Kigali road deals in three items namely fabrics, plastics and electronics. On a certain day, the sales made were 30% fabrics, 50% plastics and the rest were electronics. The shop registered a profit on fabrics and plastics of 120 and 50 million Rwandan Francs, respectively, but made a loss of 80 million Rwandan Francs on electronics. Determine the expected profit of the shop on this day.

Solution:

The expected profile would be 45,000,000

Question 14.4

The Table shows a probability distribution.

X	8	12	16	20	24
P(X)	1/8	1/6	3/8	1/4	1/12

Required:
Find
i) E(X)
ii)E(X_2)
iii)E$(x - \bar{x})(x - \bar{x})$

Solution:
i) 16
ii) 276
iii) 20
Question 14.5
A whole sale shop on Kigali rode deals in fabrics, plastics and electronics. On a certain day, the sales made were 30% fabrics, 50% plastics and the rest were electronic .The shop registered a profit on fabrics and plastics of 120 and 50million Rwandan Francs, respectively, but made a loss of 80 millions Rwandan Francs on electronics. Determine the expected profit of the shop on this day.
Solution:
Frw 45,000,000.
Question 14.6
A box contains six mangos and four oranges. Three fruits are drawn at random and not replaced.

Required:

Find the probability distribution for the number of oranges drawn.
Solution:

X	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

Question 14.7

Given A and B are two independent random variables with E(x) = 0.4 E(y) = 0.7 and Var (x) = 0.2 Var(y) = 0.4.
Required:
Find
a) $E(2x + 3y)$
b) $Var(x - y)$

Solution:
a) 2.9
b) 0.5

INTRODUCTION TO TECHNIQUES OF COUNTING

15.1. Study Objectives

By the end of the chapter, you should be able to:

- Define a permutation and a combination;
- differentiate between problems to deal with permutation and combinations; and
- Solve problems involving permutation and combinations.

This chapter deals with approaches to problems involving arrangements and selections. Combinations and permutations are smaller grouping of objects often selected from a larger population. Objects in a combination are selected simultaneously from the population. Objects in a permutation are arranged sequentially from the population.

15.2. Factorial notation

n! ('n factorial'.) is defined as the product of all integers from **n** to **1** inclusive, that is:
n! = n(n-1)(n -2)(n -3)...(2)(1)
Note: 0! = 1 and 1! = 1.

Example 15.1
Find 5!
Solution:

5! = 5(5-1)(5-2)(5-3)(5-4)
= 5x4x3x2x1
= 120

Note: n must be a positive whole number.

15.3. Arrangements

This deals with finding the number of ways in which given objects can be re arranged. The number of ways of n distinct items arranged in a row can be obtained as n

Example 15.2

In how many ways can the three letters of the words **RAT** be rearranged?
Solution:

RAT can be rearranged as RAT, ATR, TAR, RTA, ART,TRA, So, we get six different ways of re arranging the word **RAT.** There are three distinct letters of the word RAT, n=3.
∴Number of ways the letters of the word RAT can be arranged 3!=3x2x1=6.

Example 15.3

A transport manager has to plan routes for his drivers. There are four (4) deliveries to be made to customers X, Y, W and Z on different routes. How many routes can be followed?
Solution:

In this case the distinct routes to be followed, n=4.

∴Number of ways the routes can be followed =41=4x3x2x1=24.

The possible routes can also **be** listed as shown below:
XYWZ, XWYZ, XZWY, XZYW. XYZW, XWZY
Y XWZ, YXZW, YWXZ, YWZX, YZXW, YZWX
WXYZ, WXZY, WYZX, WYXZ, WZXY, WZYX
ZWXY, ZWYX, ZXWY, ZXYW, ZYWX, ZYXW

Therefore, 24 routes.
Note: As n becomes larger the listing becomes more tedious therefore use of n! becomes important.

15.4. Permutation

This is an arrangement of n distinct items in a specific order. AB and BA are different arrangements. These arrangements are known as permutations.
The number of possible arrangements when r objects are taken from a total of n distinct

objects is given by the permutation formula $^{n}Pr = \frac{n! \quad n!}{(n-r)!(n-r)!}$

Example 15.4

A company has four training officers A, B, C and D. It is required to assign one to each of the two training sections X and Y. In how many different ways may the four officers be assigned to the two sections?

Solution:

$^{n}Pr = \frac{n! \quad n!}{(n-r)!(n-r)!}$

In this case n= 4 officers and r=2 sections

$^{4}P_{2} = \frac{4! \quad 4!}{(4-2)!(4-2)!} = \frac{4x3x2x1}{2x1} = 12 \text{ ways} \frac{4x3x2x1}{2x1} = 12 \text{ ways}$

Example 15.5

In how many ways can 7 books be arranged on a shelf if taken from *10* books?

Solution:

This can be done in $^{10}P_{7} = \frac{10! \quad 10!}{(10-7)!(10-7)!} = \frac{10x9x8x7x64x3x2x110x9x8x7x64x3x2x1}{3x2x1 \qquad 3x2x1}$=**604800** ways

Permutation with identical items

One set of identical items

The number of permutations of n items taken n at a time, when r of the items are identical and the rest are all different is $\frac{n!n!}{r!r!}$

Example 15.6

In how many ways can the letters of the word KIGALI be arranged?

Solution:

There are 7 letters, so n =7. A appears three times, so r =3.

Number of permutations $\frac{7!}{3!} = \frac{7x6x5x4x3x2x1}{3x2x1} = 7x6x5x4 = \mathbf{840}$

Two sets of identical items

The number of permutations of n items taken n at a time, when r items are identical and of one kind, q items are identical and of a second kind and the rest are all different is $\frac{n!\ \ n!}{r!q!r!q!}$

Example15. 7

Find the number of permutations of the letters of the word PARALLEL.

Solution:

There are 8 letters, so n = 8, the A appears two times, so r=2, L appears three times so q=3. Number of permutations

$=\frac{8!}{2!3!} = \frac{8x7x6x5x4x3x2x1}{(2x1)(3x2x1)} 8x7x6x5x2 = \mathbf{3360} \frac{8!}{2!3!} = \frac{8x7x6x5x4x3x2x1}{(2x1)(3x2x1)} 8x7x6x5x2 = \mathbf{3360}$

Combinations

A Combination is a selection of items where arrangement is not important. Different permutations of the same items count as one combination.

For instance, there is only one combination of the letters ABC.

The number of combinations for r objects chosen from n is written as ${}^nC_r$. This number is given

By ${}^nC_r = \frac{nprnpr}{r!\ \ r!} = \frac{n!\ \ \ \ n!}{(n-r)!r!(n-r)!r!}$

In some cases, the notion $\binom{n}{r}\binom{n}{r}$= ${}^nCr = \frac{npr!npr!}{r!\ \ r!} = \frac{n!\ \ \ \ n!}{(n-r)!r!(n-r)!r!}$ is used

Example 15.8

A team is made of five members. Three members are to be chosen to represent a team at a conference.

Required:

Find the number of ways the three can be chosen.

Solution:

${}^5C_3 = \frac{5!\ \ \ \ 5!}{(5-3)!3!(5-3)!3!} = \frac{5!}{3!2!} = \frac{5x4x3x2x1\ \ \ 5!}{(3x2x1)(2x1)3!2!} = \frac{5x4x3x2x1}{(3x2x1)(2x1)} = 10$

Note:

- ${}^nC_1$=**n,** for any positive whole number n. Thus ${}^5Ci$ = 5.
- ${}^nC_1$ = 1, for any positive number n. Thus ${}^4Co$ = 1.

Example *15.***9**

A sub-committee of six, including a chairperson, is to be chosen from a main committee of twelve. If the chairperson is to be a special member of the main committee, in how many ways can the sub-committees be chosen?

Solution :

As the specified member of the main committee has to be included, we require the number of combinations of 5 from 11.

$$\binom{11}{5} = \frac{11!}{(11-5)!\,5!} = \frac{11x10x9x8x7x6x5x4x3x2x1}{5x4x3x2x1} = 462$$

Example 15.10

Find the number of ways a committee of four people can be chosen from a group of five men and seven women when it contains:

i) only men;
ii) only people of the same sex;
iii) one woman and three men; or
iv) people of both sexes and there are at least as many women as men.

Solution:

i) An all-male committee can be chosen in $^5C4=\frac{5!\quad 5!}{4!1!4!1!}$=5 ways

ii) An all-female committee can be chosen in $^7C_4\frac{7!\quad 7!}{4!3!4!3!}=\frac{7x6x57x6x5}{1x2x31x2x3}$=35 ways

There are 5 + 35 = 40 ways of choosing a committee where all the members are of the same sex.

i) One woman and three men can be chosen in $^7C_1x^5\mathbf{C_3}=\frac{7!}{\text{[illegible]}}x\frac{5!}{\text{[illegible]}}\frac{7!}{\text{[illegible]}}x\frac{5!}{\text{[illegible]}}$=7x10=70

ii) The committee could consist of two men, two women or one man, three women (four women is not allowed because people of both sexes must be included).

The number of ways to have two men and two women is

$^5C_2x^7\mathbf{C_2}=\frac{5!}{3!4!}x\frac{7!}{4!3!}\frac{5!}{3!4!}x\frac{7!}{4!3!}$=10x21=210

The number of ways to have one man and three women **is**

$^5C_2x^7\mathbf{C_3}=\frac{5!}{4!1!}x\frac{7!}{4!3!}\frac{5!}{4!1!}x\frac{7!}{4!3!}$=5x35=175

Therefore, the total number of ways **is** 210 + 175 = 385

Application

Permutations and combinations are used in solving probability problems.

Example 15.**11**

Three vacant jobs are to be filled.

Required:

Find:

- The number of ways of arranging the three vacant jobs between five men, where any man can do only one job?
- The probability that a man A will be doing job 1?

Solution:

- We require a number of permutations since putting ABC shows that man A does job 1, man B does job 2, etc., it is clear that different arrangements of letters mean different men to different jobs

Thus, number of arrangements possible $^5P_3 = \frac{5!}{2!} = \frac{5!}{2!} = 6060$

- First, we need the number of permutation that satisfies AXX, where X means any man except A:

The X's can be filled by any 2 of the other 4 men and thus done in,

$^4P_2 = \frac{4!}{2!} = \frac{4!}{2!} = 12 \text{ ways} 12 \text{ ways}$

Therefore, Pr (man A does job 1)= $\frac{n(\text{man A does job 1})}{n(\text{arranging jobs between this man})} = \frac{n(\text{man A does job 1})}{n(\text{arranging jobs between this man})}$

$= \frac{1212}{6060} = \mathbf{0.2}$

Example 15.12

A committee of four must be chosen from three women and four men.

Required: Calculate;

a) Number of ways:

i) the committee can be chosen; and

ii) Two men and two women can be chosen on the committee?

b) Probability that committee consists of:

i) two men and two women: and

ii) at least two women

Solution:

It is only selections. That is why we are using combinations.

a) i) We need to choose four people from seven. This can be done in

$^7C_4 = \frac{7!}{4!3!} = 35 \text{ ways} \frac{7!}{4!3!} = 35 \text{ ways}$

ii) With two men to be chosen from four men, can be selected in $^4C2 = \frac{4!}{2!2!} = 6 \text{ ways} \frac{4!}{2!2!} = 6 \text{ ways}$.

Similarly, two women can be selected from three women in $^3C_2$= 3 ways. Thus, they can both be Selected in a total of 6x3 = 18ways.

b) i) (two men and two women on committee)= $\frac{n(\text{2men and 2 women})}{n(\text{any 4 people})} = \frac{18}{3} \frac{n(\text{2men and 2 women})}{n(\text{any 4 people})} = \frac{18}{3}$

ii)P (committee includes at least two women) =

P (two men and two women or one man and three women) =

P (two men and two women) + P (one man and three women) *mutually exclusive events.* $(^4C_2 x ^3C_2) \div (^7C_4) + (^4C_1 x ^3C_3) \div (^7C_4) = \frac{2222}{3535}$

Self-test questions

Question 15.1

Evaluate:

a) 5! ;

b) 2! X 3! ;

c) $\frac{6!6!}{3!3!}$;And

d) $\frac{10!10!}{6!4!6!4!}$

Solution:

a) 120

b) 210

c) 120
d) 210

Question 15.2
Evaluate:

a) ^5C_l;and
b) $^6P_2$

Solution:
a) 5
b) 30

Question 15.3

If a four-digit number is formed from the digits 1, 2, 3, and 5and repetitions are not allowed, find the probability that the number is divisible by 5.

Solution:

The answer is 1/4%.

Question 15.4

The letters of the word MATHEMATICS are written, one on each of 11 separate cards. The * cards are laid out in a line.

a) calculate the number of different arrangements of these letters.
b) Determine the probability that the vowels are all placed together.

Solution:

a) $\frac{11!}{2!2!2!}$a) $\frac{11!}{2!2!2!}$=4,989,600

b) Arranging letters without vowels gives $\frac{8!\ 8!}{2!2!2!}$=10,080
The vowels A, E, A, I however can be arranged in 12 ways. So, total number of arrangement =12x10,080=120,960
P (vowels together) =$\frac{120960\ 120960}{4989600\,4989600}$=0.024(2sf)

Question 15.5

Seven cards labelled A, B, C, D, E, F, G are thoroughly shuffled and then dealt out face upwards on a table. Find the probability, giving each as a fraction in its simplest form, that:

a) the first three cards to appear are the cards labeled A, B, C in that order.
b) the first three cards to appear are the cards labeled A, B, C, but in any order.
c) the seven cards appear in their original order: A, B, C, D, E. F, G.

Solution:

a) $^7\mathbf{P}_3$ **=**$\frac{7!7!}{4!4!}$=7x6x5=**210**
Therefore, P (first three cards labeled A, B, C) =$\frac{1\quad 1}{210210}$

b) P (any order)=$\frac{1\quad 1}{3535}$

c) No of ways=5040

Therefore P(A,B,C,D,E,F,G) =$\frac{1\quad 1}{50405040}$

Question 15.6

The ICPAR wants to open five centres in twelve districts of Rwanda . Each new centre will be in a different district. The centres will be administering exams on the same day, same time. How many different ways can these five centres be situated among the twelve possible districts of Rwanda ?

Solution:

$$^{n}C_{x}=\frac{n7!\quad n7!}{(n-x)x!(n-x)x!}=$$

$$=^{12}C_{6}=\frac{12!}{(12-5)5!}=792\frac{12!}{(12-5)5!}=792$$

Question 15.7

From a Sacco of ten men and eight women, a committee is formed. The committee will have five men and four women. How many different ways can this committee of five men and four women be formed from the men and women above, if all committee members are picked at the same time.

Solution:

$$^{n}C_{x}=\frac{n!\quad n!}{(n-x)x!(n-x)x!}$$

$$=^{10}C_{5}x^{8}C_{4}$$

$$=\frac{10!}{(8-5)4!}x\frac{8!\quad 10!}{(8-4)4!(8-5)4!}x\frac{8!}{(8-4)4!}=252x70=17640$$

Question 15.8

How many ways can seven job seekers be seated on a waiting bench if only four seats are available and the four seats are filled sequentially by the available seven job seekers?

Solution:

$$^{n}P_{x}=\frac{n!\quad n!}{(n-x)x!(n-x)x!}=$$

$$^{7}P_{4}x^{8}C_{4}=\frac{7!}{(7-4)!}=360\text{ ways}\frac{7!}{(7-4)!}=360\text{ ways}$$

Question 15.9

Five people of whom three are women and two are men, are to form a queue. Find how many different arrangements there are i) if no two people of the same sex are to stand next to each other, ii) if the first and last people in the queue are both to be men.

Solution:

i) 12
ii) 12

Question 15.10

A sub-committee of six, including a chairperson, is to be chosen from a main committee of twelve. If the chairperson is to be a specified member of the main committee, in how many ways can the sub-committee be chosen?

Solution:

The answer is 462.

CHAP 16.

BINOMIAL AN DPOISSON DISTRIBUTION

16.1. Study objectives

By the end of this chapter, you should be able to:

- Define a binomial distribution;
- state the uses of a binomial distribution;
- list the properties of binomial distribution;
- give examples of a binomial distribution;
- Calculate probabilities of a binomial distribution using the formula;
- use mathematical tables to find cumulative binomial probabilities; and
- find the expectation, variance and standard deviation of a binomial distribution

16.2. Binomial distribution

Binomial definition and examples

A binomial distribution is used whenever a process has only two possible outcomes referred to as either success or failure. For instance:

i) candidates who sit an examination may either pass or fail the examination;
ii) number of loan holders in a certain period may either pay on time or not on time;
iii) a multiple-choice question, the answer given may be classified as either correct or incorrect;
iv) when a baby is born, it may be either male or female;
v) number of items from a production line may be faulty or not; and
vi) When a coin is tossed, it may land either head or tail.

Note: One of the possible outcomes is a success while the other is a failure.

Properties of a binomial experiment

A binomial experiment or a Bernoulli trial is a probability experiment that has the following properties:

- There must be number of repeated identical trials, n.
- Each trial results into two possible outcomes referred to as either success or failure.
- The trials are independent i.e. the outcome on one trial does not affect the outcome on the other trial.
- The probability of success is the same on every trial.

Note: The probability of success is denoted by p while the probability of failure is denoted by

q Where q= 1-P.

Consider the binomial experiment, of tossing a coin three times and the number of times the head shows on the coin is counted.

In such an experiment the following observations are made:

j) The experiment consists of fixed identical (repeated) trials, n =3.

ii) Each trial results into two possible outcomes, that is, either getting a head or not getting a head.

iii) The probability of success is the same on every trial, that is, P(getting a head on the first trial) =P(getting a head on the second trial) = P(getting a head on the second trial) = 1/2

iii) The trials are independent, that is, getting a head on one trial does not affect getting a head on other trials.

Binomial probabilities

A binomial distribution is obtained by the outcomes of a binomial experiment with their corresponding probabilities.

The binomial probability refers to the probability that the binomial experiment results in exactly r successes.

If the parameters r, n and $_P$ are given then the binomial probabilities can be computed using the formula:

$b(r, n, p) = P(X = r) = \binom{n}{r}\binom{n}{r} p^r q^{n-r}$ where $= \binom{n}{rp}\binom{n}{rp} = \frac{n! \quad n!}{(n-r)!(n-r)!}$

Where r is the number of successes that result from the binomial experiment

n is the number of repeated trials in the binomial experiment

p is the probability of success on an individual trial

q is the probability of failure and q = 1 –p

b used as the short form for binomial

P(X = r)

This is the probability distribution function in which X is the binomial random variable defined as the number of successes r in n independent repeated trials of a binomial experiment.

The binomial random variable is denoted as x ~B(n,p)

where x is the binomial random variable

the symbol ~ means “has distribution”

ββ is used as the short form for binomial

n and p are parameters **of** the distribution; n number of independent trial and **p** probability of success

a) Using the formula to find the binomial probabilities

Example 16.1

In a survey. 40% of the people in the working class interviewed said that they bought newspapers only over the weekend. If ten people are selected at random,

Required:

Find the probability that exactly four of these people buy the newspapers only over the weekend.

Solution:

p=40%=0.4, q=1-p = 1- 0.4=0.6. n=10

Let X be the random variable that a person buys newspapers only over the weekend

∴X~B(10,0.4)

$P(X = 4) = C_4^{10})0.4^4\ 0.6^{10-4}$(use a calculator to compute)

=210X0.02S6X0.0467

= 0.2511 (4dp)

Example 16.2

A medical research shows that of 12 patients randomly selected suffer from a rare disease, 68 % die of it. Required:

Find the probability that fewer than three patients will recover?

Solution:

We are given 68% die implying that 32% will recover in this case p=0.32 n =12, p=0.32 and q = 0.68

Let X be the random variable that a patient recovers .-. X~B(12, 0.32)

P(X < 3) = P(X = O) + P(X = 1) + P(X =2)

$=\binom{12}{0}0.32^{0}0.68^{12}\binom{12}{0}0.32^{0}0.68^{12}\quad+\binom{12}{1}0.32^{1}0.68^{11}\binom{12}{1}0.32^{1}0.68^{11}\quad+\binom{12}{2}0.32^{2}0.68^{10}\binom{12}{2}0.32^{2}0.68^{10}$

=0.0098+0.0552+0.1429

=0.2079(4dp)

Example 16.3

The probability that a student is able to pass a financial accounting test is 0.20 if seven students are selected at random

Find the probability that:

i) fewer than 3 are able to pass
ii) exactly three are able to pass
iii) more than 3 are able to pass
iv) at least three are able to pass

solution :

given that x$\sim B(7,0.20)\sim B(7,0.20)$

i)p (x **< 3)** = P(x = 0) + p(x **= 1) +** p (X **= 2)**

$=\binom{7}{0}0.20^{0}0.8^{7-0}=\binom{7}{0}0.20^{0}0.8^{7-0}\quad+\binom{7}{1}0.20^{1}0.8^{7-1}\binom{7}{1}0.20^{1}0.8^{7-1}\quad+\binom{7}{2}0.20^{2}0.8^{7-2}\binom{7}{2}0.20^{2}0.8^{7-2}$

=0.2097+0.3670+0.2753=0.8520

ii)$P(x=3)=\binom{7}{3}0.20^{1}0.8^{7-3}=0.1147P(x=3)=\binom{7}{3}0.20^{1}0.8^{7-3}=0.1147$

iii)P(x>3)= $P(x=4)+P(x=5)+P(x=6)+P(x=7)\,P(x=4)+P(x=5)+P(x=6)+P(x=7)$

$P(x>3)=\binom{7}{4}0.20^{4}0.8^{7-4}P(x>3)=\binom{7}{4}0.20^{4}0.8^{7-4}+\binom{7}{5}0.20^{5}0.8^{7-5}\binom{7}{5}0.20^{5}0.8^{7-5}\quad+\binom{7}{6}0.20^{6}0.8^{7-6}$
$\binom{7}{6}0.20^{6}0.8^{7-6}\quad+\binom{7}{7}0.20^{7}0.8^{7-7}\binom{7}{7}0.20^{7}0.8^{7-7}$

$P(x>3)=0.0287+0.0047+0.0004+0=0.0334$

$P(x\geq 3)P(x\geq 3)$=0.1147+0.0334

=0.1481

Example 16.4

Given that x **~s(7,0.7) find** i) p (x $\leq\leq$ **4) ii**) p{x $\geq\geq$ **4)**

Solution:

i)P(x≤≤,**4)=** p(x = **o)** **+** p(x = **1)** **+** p (x = 2) + p(x = 3) + p (x = **4)**

$=\binom{7}{0}0.70^0 0.3^{7-0}\binom{7}{0}0.70^0 0.3^{7-0}$ $+\binom{7}{1}0.70^1 0.3^{7-1}\binom{7}{1}0.70^1 0.3^{7-1}$ $+\binom{7}{2}0.70^2 0.3^{7-2}\binom{7}{2}0.70^2 0.3^{7-2}$ +

$\binom{7}{3}0.70^3 0.3^{7-3}\binom{7}{3}0.70^3 0.3^{7-3}$ $+\binom{7}{4}0.70^4 0.3^{7-4}\binom{7}{4}0.70^4 0.3^{7-4}$

=0.0002+0.0036+0.0250+0.0972+0.2269

=0.3529

ii)P(x ≥ 4) =P(x =4)+P(x=5)+P(x=6)+P(x=7)

$=\binom{7}{0}0.70^0 0.03^{7-0}\binom{7}{0}0.70^0 0.03^{7-0}$ $+\binom{7}{1}0.70^1 0.03^{7-1}\binom{7}{1}0.70^1 0.03^{7-1}$ $+\binom{7}{2}0.70^2 0.03^{7-2}\binom{7}{2}0.70^2 0.03^{7-2}$

$+\binom{7}{3}0.70^3 0.03^{7-3}\binom{7}{3}0.70^3 0.03^{7-3}$ $+\binom{7}{4}0.70^4 0.03^{7-4}\binom{7}{4}0.70^4 0.03^{7-4}+\binom{7}{5}0.70^5 0.03^{7-5}\binom{7}{5}0.70^5 0.03^{7-5}$ +

$\binom{7}{6}0.70^6 0.03^{7-6}\binom{7}{6}0.70^6 0.03^{7-6}$ $+\binom{7}{7}0.70^7 0.03^{7-7}\binom{7}{7}0.70^7 0.03^{7-7}$

0.2269+0.3177+0.2471+0.0824=0.8741

Example16. 5

A coin is tossed eight times. Required:

Find the probability that:

i) Exactly three heads show up.

ii) At least five heads show up.

iii) More than one but less or equal to four heads show up.

Solution:

n =8 and p=0.5 q = 1 - 0.5 = 0.5

Let X denote the number of heads that show up:

X~B (8, 0.5)

i) P(X = 3) = $=\binom{8}{3}0.5^3 0.5^{8-3}\binom{8}{3}0.5^3 0.5^{8-3}$=0.2188(4dp)

ii) P(x ≥≥5) = p(x = 5) + p(x= 6) + p(x= 7) + p(x = 8)

$\binom{8}{2}0.5^2 0.5^6\binom{8}{2}0.5^2 0.5^6+\binom{8}{3}0.5^3 0.5^3\binom{8}{3}0.5^3 0.5^3+\binom{8}{4}0.5^4 0.5^4\binom{8}{4}0.5^4 0.5^4=$

=0.1094+0.2188+0.2734

=0.6016(4dp)

Example 16. 6

A factory produces spare parts of vehicles and the quality controller finds out that 15% of the spare parts are defective.

Required:

Find the probability that a batch of 15 spare parts will contain:

a) No more than four are defective; and

b) at least four are defective.

Solution:

Let X be the number of defective spare parts, in this case success means being defective X - B(15, 0.15), n=15, p=0.15 q=0.85

a) p(x ≤≤ **4)** **=** p(x **= o)** **+** p(x **= 1)** **+** p(x **= 2)** **+** p(x **= 3)** **+** p(x **= 4)**

$=\binom{18}{0}0.15^0 0.85^{15}\binom{18}{0}0.15^0 0.85^{15}$ $+\binom{18}{1}0.15^1 0.85^{14}\binom{18}{1}0.15^1 0.85^{14}$ $+\binom{18}{2}0.15^2 0.85^{13}\binom{18}{2}0.15^2 0.85^{13}$ +

$\binom{18}{3}0.15^3 0.85^{12}\binom{18}{3}0.15^3 0.85^{12}$ $+\binom{18}{4}0.15^4 0.85^{11}\binom{18}{4}0.15^4 0.85^{11}$

=0.0871+0.2312+0.2856+0.2184+0.1156

=0.9382(4dp)

b) P(X>4) = 1 - P(X<4)
= **1 -** [P(X **= 0)** + P(X **= 1)** + P(X **= 2)** + P{X **= 3)]** = 1 - [0.0874 + 0.2312 + 0.2856 + 0.2184]
= 1 - **0**.8226
= 0.1774(4dp)

Example 16.7

If 5% of the children immunized against polio get it after immunization.
Required:
Find the probability that for a sample of nine children immunized:
at most two got polio; and
more than two got polio.
Solution:

n =9, p=5%=0.05, q=0.95
Let X be the binomial random variable that a child gets Polio after immunisation

a) P(X ≤≤ 2) = P(X =0) + P(X=1) +P(X=2)
= (2)0.05^0 0.95^9 +Q$0.05^1 0.95^8$ +Qo.$05^2 0.95^7$

= 0.6302+0.2985+0.0629 = 0.9916

b) P(X >2) =1 - P(X ≤≤ 2)
= 1 -0.9916
= 0.0084

Use of tables to compute cumulative probabilities for the binomial distribution

a) Cumulative binomial probability

This refers to the probability that the binomial random variable falls within a specified range, that is, when finding probabilities involving either x ≥ to the stated lower limit or x ≤≤ to the stated upper limit.
Before you use any cumulative binomial probability tables one has to understand how to read the tables.
There are tables that give either P(X ≥≥r) or P(X ≤≤r).
To use the cumulative binomial probabilities tables using P(X ≤≤ r)

We try to express the probability that we are looking for in terms of P(X< r) using the relations in the table.
The different key words that can be used to ask for probabilities and how to relate with the tables.

Key words	Mathematical symbol	Relationship with p(x≤≤ r)
P (X is fewer than r) e.g., P(X is fewer than 4)	p(x < r)e.g. p(x <4)	P(X≤≤r-I) e.g. P(X ≤≤3)
P(X is not more than r) e.g., P(X is not more than 6)	p(x ≤≤r) e.g. p(x≤≤6)	P(X ≤≤6)

P(X is exactly r) e.g., P(X is exactly 5)	p(x = r) e.g. PC x = 5)	P(X≤≤r)-P(X≤≤r-I) P(X≤≤5) — P{X≤≤4)
P(X is at least r) e.g., P(X is at least 8)	p(x ≥≥r) e.g. p(x ≥≥8)	1-P(X≤≤r-I)$_e$.g. 1 –P(X ≤≤7)
P(X is greater than r) e.g. P(X is greater than 10)	p (x ≥≥r) e.g. p(x≥≥10)	I-P(X ≤≤r)eg. 1 - P(X ≤≤10)
P(X is at most r) e.g., P(X is at most 2)	P(X ≤≤r)e.g.P(X ≤≤2) P(X ≤ ≤2)	P(x ≤ 2)

Example16. 8

The probability that a student is able to pass a financial accounting test is **0.20. If** seven students are selected at random.

Required:
Find the probability that:
i) fewer than 3 are able to pass.
ii) exactly three are able to pass.
iii) more than 3 are able to pass.
iv) at least three are able to pass.

Solution:

Given that x ~ B (7,0.20**)**

Extract from the table for n = 7 and p = **0.20**

Because the table gives the values of P(X < r) then we express the probabilities asked in terms of P(X < r)

i) P (x <**3)** =P(x ≤≤**2) = 0.8520** (under n =7 look up the value at the intersection of the row of r=2 and column of p=0.20).

ii)P(X **= 3) = P(x > 3) –** P(X≤≤**2)**

= 0.9667 — 0.8S20

= 0.1147

iii) P(X ≥≥ 3) = 1 - P(X ≤≤3)

= 1 - 0.9667

= 0.0333

iv)P(X≥≥3) = 1 - P(AT ≤≤ 2)

= 1 - 0.820

= 0.1480

Example 16.9

Given that x ~B(7,0.7) find i)P(x≤≤4) ii) P(x≤ **2**)≤ **2**)

Note that the values of p stop at 0.50, in the question above p = 0.70 which is greater than 0.50, so the probability of success changes to 0.30 and the variable also changes such that Y ~B (7, 0.3)

i) P (X ≤≤4) = P(Y ≥≥7- 4) = P(y≥≥3) = 1 - P(Y ≤≤ 2) = 1 - 0.6471 = 0.3529

ii) P(x≥≥4) = P(Y ≤≤7-4) = P(Y ≤≤3) = 0.8740

Note: In case the value of p is not found in the table then it is advisable you use the binomial formula

1. The expectation and variance of a binomial distribution

- The expectation (mean or expected value) of a binomial random variable X is cover by the formula
  E(X) = np

- The variance of a binomial random variable X is given by the formula
  Var(X) = npq where q = 1 — p

- The standard deviation of a binomial random variable X, $\delta\delta=\sqrt{\mathrm{var}(x)} = \sqrt{npq}\sqrt{\mathrm{var}(x)} = \sqrt{npq}$

Example 16.10

Given that x ~B(100,0.75)

Required:

Find the.

a) E(X); and
b) Standard deviation of X.

Solution

x ~B(100,0.75) n=100, p=0.75 and q =1-0.75=0.25

a) E(X)=np=100x 0.75=75
b) $\sigma\sigma = \sqrt{npq} = \sqrt{100x0.75x0.25}\sqrt{npq} = \sqrt{100x0.75x0.25}$= 4.330

Example 16.11

During routine maintenance, it is found out that two in every five components of a machine are faulty. In order to confirm this, another survey is carried out and ten machines are chosen at random. The random variable X represents the number of machines whose components are faulty.

Required:

Determine the:

a) mean of X;
b) variance of X; and
c) Standard deviation of X.

Solution:

a) X-B(n.p) with n = 10, p = | = 0.4, q = 1 - 0.4 = 0.6

   E(X) = np=10x0.4 =4

b) Var(X) = npq =10x0.4x0.6 =2.4

c) $\sigma\sigma= \sqrt{npq}\sqrt{npq}= \sqrt{2.4}\sqrt{2.4}$ = 1.5492

Example 16.12

The random variable X follows a binomial distribution, given that E(X) = 4.8 and p=0.6. **Required:** Find the:

a) value of n; and
b) Var(X).

Solution:

a) E(X) = np

4.8 = n x 0.6 4.8

= n

0.6

n=8

b) Var(X) = npq= 8 x 0.6x 0.4 = 1.92

Normal approximation to the binomial distribution

If a binomial distribution has parameters n and p, the distribution can be approximated by a normal distribution with mean np and variance npq when the value of:

- n is large (n 50)
- np 5 and nq 5

Example 16.13

36% of the 7,000 employees of mineral water manufacturing industry belong to unions. The personnel manager of the company takes a sample of 100 employees in order to find out union members' attitude towards management. He considers the selection of a union member in his sample a success.

Required:

a) Verify that the distribution can be approximated by a normal distribution.

b) Use the normal approximation to calculate the probability that the number of successes will be between 24 and 42

Solution

a) =np=100 0.36=36 and nq=100 0.64=64

Since np>5 and nq>5, we use a normal approximation with mean= np=36

σσ= $\sqrt{100x0.36x0.64}$ $\sqrt{100x0.36x0.64}$ =$\sqrt{23.04}$ $\sqrt{23.04}$ =4.8

The standardized value of X=24 when =36 and =4.8 is

Z= = =-2.5

The standardised value of X=42 is

$$Z = \frac{(x-\mu)}{-} = \frac{(42-36)}{42} = 1.25$$

P(24 < X < 42) « P(-2.5 < X < 1.25) = .4938 + .3944 = .8882

Note: The exact probability from the binomial tables is 0.9074

Use of Poisson approximation for a binomial distribution

The conditions for the Poisson distribution to be used to approximate binomial distributions are:

- The n must be large (n 50)

- The value of p (probability of success) must be close to zero (p   0.1)

When these two conditions hold, the Poisson distribution with mean u=np can be used to approximate binomial probabilities.

Example 16.14

It is estimated that on average one person in every 1,000 makes a numerical error in preparing an income tax return. Eight thousand forms are selected at random and examined.

Required:

a) Verify that the distribution can be approximated by a Poisson distribution.
b) Calculate the probability that the sample will yield less than seven errors

Solution

a) The experiment is a binomial with n=8,000 and p=0.001 and since n is very large and p is very close to zero, Poisson approximation is used.
b) Using u= 8,000x 0.001=8

P(X <7) =b(x; 8000,0.001)

$\approx \sum^{6} p(x,8)\sum^{6} p(x,8)$

Where x =0,1,2,3,4….

16.3. Poisson distribution

Poisson distribution is named after a French mathematician Simeon D. Poisson (1781-1840). This is another type of discrete probability distribution used when n is large (i.e., n > 30), p is small and when the independent variables occur over a period of time.

A discrete random variable X is said to follow a Poisson distribution whose probabilities can be calculated using the formula:

P(X = x) = P(x,$\lambda\lambda$) = $\frac{e^{-\lambda}\lambda^{x} e^{-\lambda}\lambda^{x}}{x! \quad x!}$

Where X = 0, 1,2,3,4,

$\lambda\lambda$ is the mean number of occurrences,

e is a constant approximately equal to 2.7183

If a random variable X, follows a Poisson distribution then X ~ Po(□) where Po is short form for Poisson.

Events that may follow a Poisson distribution

- Car accidents on a particular road in one day.
- Telephone calls made to a switch board in a given minute.
- Accidents in a factory in one week.
- Insurance claims made to a company in a given time.

Example 16. 15

In a batch of 1,000 computers, the mean of the defective computers is ten. If a random sample of 120 is selected.

Required

Find the probability of:

a) exactly six defective ones: and

b) at most two defective ones.

Solution:

λλ=10:x≈≈p(x:10)

a)P(6,10)=$\frac{e^{-\lambda}\gamma_{\gamma}\lambda^{x}}{x!} = \frac{e^{-10}10^{6}}{6!}\frac{e^{-\lambda}\gamma_{\gamma}\lambda^{x}}{x!} = \frac{e^{-10}10^{6}}{6!} = \frac{2.7183^{-10}10^{6}}{6!}\frac{2.7183^{-10}10^{6}}{6!}$=0.0631

b) **P**(X$\leq 2) = P(X = 0) + P(X = O) + P(X = 1) + P(X = 2)$
$\leq 2) = P(X = 0) + P(X = O) + P(X = 1) + P(X = 2)$

=$\frac{e^{-10}10^{6}}{0!}\frac{e^{-10}10^{6}}{0!}+\frac{e^{-10}10^{16}}{1!} + \frac{e^{-10}10^{2}}{2!}\frac{e^{-10}10^{16}}{1!} + \frac{e^{-10}10^{2}}{2!}$

$\frac{2.7183^{-10}10^{0}}{0!} + \frac{2.7183^{-10}10^{1}}{1!} + \frac{2.7183^{-10}10^{2}}{2!}$

2.469X$10^{-3=0.00277(5dp)}10^{-3=0.00277(5dp)}$

Example 16. **16**

On average, a human resource manager receives eight telephone calls between 8:30am and 9:30am on a particular day.

Required:

Find the probability that she will receive two or more calls between 8:30am and 9:30am on a particular day.

Solution

λλ=8..x⋯⋯p_0(8)

P(X$\geq 2) = 1 - [P(X) = 0) + P(X = 1)]\geq 2) = 1 - [P(X) = 0) + P(X = 1)]$

=1-$\left|\frac{e^{-8}8^{0}}{0!} + \frac{e^{-8}8^{1}}{10!}\right|\left|\frac{e^{-8}8^{0}}{0!} + \frac{e^{-8}8^{1}}{10!}\right|$

=1-$\left|\frac{2.7183^{-8}8^{0}}{0!} + \frac{2.7183^{-8}8^{1}}{1!}\right|\left|\frac{2.7183^{-8}8^{0}}{0!} + \frac{2.7183^{-8}8^{1}}{1!}\right|$

=1-$[3.354X10^{4} + 2.684X10^{3}][3.354X10^{4} + 2.684X10^{3}]$

=1-0.0030194=0.9970

Example 16. **17**

A certain farm receives four calls per hour on average. Find the probability that the firm will receive the following for any given hour.

a) Exactly four calls.

b) At most two calls.

c) At least four calls.

Solution:
sometime it is easier to use the mathematic tables instead of the formula, as show below

in this question γγ=4, so we have an extract below

X	2.3 2.4 2.5 3.0 3.5	□4.0	4.5 5.0 5.5 6.0 6.5
0		.0183	
1		.0733	
2		.1465	
3		.1954	
4		.1954	
5			

a) P(4,4)=0.1957

at most two calls means 0,1 or 2 calls

b) P(at most two calls) = P(0,4) + P(1,4) +P(2,4)

= 0.0183 + 0.0733 + 0.1465 = 0.2381

c) at least four calls means four or more calls.

(in this case it is easier to find the probabilities of 0,1,2, and 3 calls and then subtract their answer from 1).
P(0,4)+ P(1,4) + P(2,4) + P(3,4) = 0.0183 + 0.0733 + 0.1465 + 0.1954 = 0.4335 .

-. P(at least four calls) = 1 - 0.4335 = 0.5665

a) **Expectation and variance of a Poisson distribution**

Given that X follows a Poisson distribution with mean number □ then:

- the Expectation, E(X) = □
- the Variance, Var(X) = □

Example 16.18

The number of accidents recorded by the traffic police fortnightly on a highway follows a Poisson distribution with standard deviation 3.2.

Required:

Find the probability that exactly seven accidents were recorded.

Solution:

Standard deviation =3.2 ∴**var(x}** = 3.2^2 = □= 10.24

P(X = 7) = $\frac{e^{-10.24}10.24^7}{7!}\frac{e^{-10.24}10.24^7}{7!} = \frac{2.7183^{-10.24}x10.24^7}{7!}\frac{2.7183^{-10.24}x10.24^7}{7!}$ 0.08364

b) **Using the Poisson distribution as an approximation to the binomial distribution**

The conditions for the Poisson distribution to be used to approximate binomial distributions are:

- The n must be large (n >50).
- The value of p (probability of success) must be close to zero (p < 0.1).

When these two conditions hold, the Poisson distribution with mean u=np can be used to approximate binomial probabilities.

Example 16.19

It is estimated that on average one person in every 1,000 makes a numerical error in preparing an income tax return. 8,000 forms are selected at random and examined.

Required:

a) Verify that the distribution can be approximated by a Poisson distribution.

b) Calculate the probability that the sample will yield less than seven errors.

Solution:

The experiment is a binomial with n=8000 and p=0.001 and since n is very large and p is very close to zero, Poisson approximation is used.

Using u= 8000x0.001 =8,

P(X<7) = $\sum_{x}^{6} 0b = 0b(x;8000,0.001)\sum_{x}^{6} 0b = 0b(x;8000,0.001)$

$\approx \sum p(x:8)$

= 0.3134

Self-test questions

Question 16.1

a) Define a binomial experiment.

b) Write down at least five the examples of binomial experiments.

c) Give the four properties of a binomial distribution.

Question 16.2

The probability that a student of Business Mathematics and Statistics answers a multipl choice question correctly is 0.25.

Required:

Calculate the probability of answering 14 questions correctly in 20 questions attempted.

Question 16.3

An unbiased die is thrown nine times.
Required:
Find the probability of getting exactly four sixes.

Question 16.4

60% of residents listen to CBS radio station. Sixteen residents are chosen at random.
Required:
Find the probability that at most three quarters of these 16 residents listen to CBS.
Question 16.5

85% of Rwanda n families have mobile phones. Twenty families are randomly selected.
Required:
Determine the probability that:

a) none of the 20 families have mobile phones;
b) aJI the 20 families have mobile phones;
c) fewer than 10 families have mobile phones; and
d) more than 16 families do not have mobile phones.

Question 16.6

A survey found out that 40% of public servants earn some pay from part-time jobs. If six public servants are selected at random.

Required:
i Find the probability that at least four of them earn a pay from part-time jobs.
Question 16.7

If 25% of Rwanda ns go for New Year Passover prayers at Nambole Stadium on 31 December. Ifa random sample of 2,000 Rwanda ns is selected.

Required:

Find the:

i) mean:

ii) variance;

iii) standard deviation of the number of individuals who go for the Passover prayers.

Question 16.8

It was reported that 92% of employees of a certain company use internet. If a sample of 150 is selected.
Required:
Find the
i) mean; and
ii) standard deviation of the number who use internet.

Question 16.9

A sample of 100 bodaboda cyclists is randomly selected. If the probability that a bodaboda cyclist is involved in an accident due to abuse of traffic laws is 0.65.

Required:

Find the

i) expected number of accidents; and

ii) Standard deviation of accidents.

Question 16.10

a) State the conditions under which the distribution B(n,p) can be approximated by a normal distribution.

b) The random variable X has the distribution B (25,0.38).

Required:

a) Verify that the distribution can be approximated by a normal distribution.
b) Use the normal approximation to calculate the probability that X takes the value 18.

Question 16.11

During an advertising campaign, the manufacturer 'KUKU' type of poultry feed claimed that 60% of his customers preferred 'KUKU' feeds Assuming that the manufacturer's claim is correct for the population of poultry keepers,

Required:

a) Use a normal approximation to the binomial.
b) Use a binomial distribution to calculate the probability that at least six of a random sample of eight poultry keepers prefer to buy KUKU feeds.

Binomial questions

Question 16.12

a) Define a binomial experiment.
b) Write down at least five the examples of binomial experiments.
c) Give the four properties of a binomial distribution.

Question 16.13

The probability that a student of Business Mathematics and Statistics answers a multiple-choice question correctly is 0.25.

Required:
Calculate the probability of answering 14 questions correctly in 20 questions attempted.
Solution: 2.5699×10^{-5}
Question 116.4
An unbiased die is thrown nine times.
Required:
Find the probability of getting exactly four sixes.
Solution: 0.00781
Question 16.15
60% of residents listen to CBS radio station. Sixteen residents are chosen at random.
Required:
Find the probability that at most three quarters of these 16 residents listen to CBS.
Solution:
0.93485
Question 16.16
85% of Rwanda n families have mobile phones. Twenty families are randomly selected.
Required:

Determine the probability that:

a) none of the 20 families have mobile phones;
b) all the 20 families have mobile phones;
c) fewer than 10 families have mobile phones; and
d) more than 16 families do not have mobile phones.

Solution:

a) 3.325x10
b) 0.0388
c) 0.9998
d) 0.3154

Question 16.17

A survey found out that 40% of public servants earn some pay from part-time jobs. If six public servants are selected at random.

Required:

Find the probability that at least four of them earn a pay from part-time jobs.

Solution:

0.1792

Question 16.18

If 25% of Rwanda ns go for New Year Passover prayers at Nambole Stadium on 31 December. If a random sample of 2,000 Rwanda ns is selected.

Required:

Find the:

i) mean;
ii) variance; and
iii) standard deviation of the number of individuals who go for the Passover prayers.

Solution:

i) 500
ii) 375
iii) 19.36

Question 16.19

It was reported that 92% of employees of a certain company use internet. If a sample of 150 is Selected.

Required:

Find the:

i) mean and
ii) standard deviation of the number who use internet.

Solution:

i) 138
ii) 3.323

Question 16.20

A sample of 100 bodaboda cyclists is randomly selected. If the probability that a bodaboda is nvolved in an accident due to abuse of traffic laws is 0.65.

Required:

Find the

a) expected number of accidents; and
b) standard deviation of accidents.

Solution:

i)) 65

ii) 4.77

Question 16.21

The random variable X has the distribution B (25,0.38).

Required:

a) Verify that the distribution can be approximated by a normal distribution.

b) Use the normal approximation to calculate the probability that X takes the value 1c

Solution:

a) n>50, np=25x 0.38=9.5

b) (18 - 9.5)/2.427=3.5027 ∴ P(X=18)=0.9998

Question 16.22

During an advertising campaign, the manufacturer 'KUKU' type of poultry feed claimed that 60% of his customers preferred 'KUKU' feeds. Assuming that the manufacturer's claim is correct for the population of poultry keepers,

Required:

a) Use a normal approximation to the binomial.

b) Use a binomial distribution to calculate the probability that at least six of a random sample of eight poultry keepers prefer to buy KUKU feeds.

Poisson questions

Question 16.23

A land survey company has established by its control system that the demand for mutation form is distributed by the Poisson distribution. It has been established that on average 5 forms are demanded every 10 minutes

Required:

Find the probability that exactly three forms will be demanded in every ten-minute interval.

Solution:

0.1396.

Question 16.24

A local bank has observed that demand for its teller machines is approximated by Poisson distribution. If on average 50 people demand a teller machine every 15 minutes at lunch time on Fridays.

Required: Find the probability that 30 people will arrive in a 15-minute period on any given lunch time on Friday.

Solution: 0.0007.

Question 16.25

It has been determined that on average 10 airplanes land at Entebbe international airport every 15 hours. Assuming the landings are approximated by the Poisson distribution.

Required:

What is the probability that 18 air planes will land every 30 days?

Solution: 0.0844

Question 16.26

a) Explain the assumptions that must be satisfied when a Poisson distribution adequately describes a random variable x.

b) State the conditions under which the binomial distribution approximates to the Poisson distribution.

c) Tests for defects are carried out in a textile factory on a lot comprising 400 pieces of cloth. The results of the tests are tabulated below:

No. of faults per piece	0	1	2	3	4	5	6	Total
No. of pieces	92	142	96	46	18	6	0	400

Required:

a) Show that this is approximately a Poisson distribution and calculate the frequencies on this assumption.

b) How many pieces from a sample of 1000 pieces may be expected to have 4 or more faults?

Solution:

a) Frequencies are: 95.25, 136.7, 98.06, 46.9, 16.83, 4.83, 1.16.
b) 22.8.

Question 16.27

The number of customers arriving at a teller's window at a commercial bank is Poisson distributed with a mean rate of 0.75 person per minute.

Required:

Calculate the probability that 2 or fewer customers will arrive in the next 6 minutes Solution:
0.1736.

Question 16.28

The number of telephone calls passing through the UCC switchboard has a Poisson distribution with mean A. equal to 3t, where t is the time in minutes.
Required:
Find the probability of:
a) Two calls in any one minute
b) Five calls in two minutes
c) At least one call in one minute
Solution:

a) 0.2240
b) 0.1606
c) 0.9502

Question 16.29

In a certain factory, the number of accidents occurring in a month follows a Poisson distribution with mean 4.
Required:
Find the probability that there will be at least 40 accidents during one year
Solution:
0.8901.

Question 16.30

A random variable x has a Poisson distribution with u =2. **Required:**
Use a formula to compute the:
a) mean;
b) variance; and
c) standard deviation.
Solution:
a) 2
b) 2

c) 1.414

Question 16.31

In a certain suburb, false alarms are received at the fire brigade station at a mean rate of 0.3 per day.
Required:

Find the probability that:
a) More than 7 days will pass before the next false alarm arrives?
b) Less than 2 days will pass before the next alarm arrives?
Solution:

a) 0.1225
b) 0.4512

Question 16.32

A stationary shop packs paper in reams of 500. The probability that a paper is defective is 0.003.

Required:
Use a Poisson approximation to find the probability that a ream contains 2 defective papers.
Solution: 0.251

Question 16.33

A factory selling electrical appliances packs them in boxes of 60. On average 2% of the appliances are faulty.
Required:
Use a Poisson approximation to calculate the probability of getting more than 2 defective appliances in the box.
Solution: 0.121

Question 16.34
The defects in an automated cloth-weaving process at a factory are Poisson distributed at a mean rate of 0.00025 per square metre. The process is set up to run 1000 metres of weaving.
Required:
Find A for this problem.
Calculate the probability that this process will produce 5 defects on the run.
Compute the probability that it will produce between 1 and 3 defects inclusive on this run.
Solution:
a) 2.25
b) 0.0507
C) 0.7038

THE NORMAL DISTRIBUTION

17.1. Study objectives

By the end of this chapter, you should be able to:

- identify properties of a normal distribution curve;
- determine the Z score;
- find areas under the standard normal distribution from the normal statistical tables: and
- find the probabilities for a normally distributed variable by transforming it into a standardised normal variable.

17.2. Normal distribution

A normal distribution is the name given to a type of distribution of continuous data that is common in practice, to distributions with a mean of 0 and standard deviation of 1.

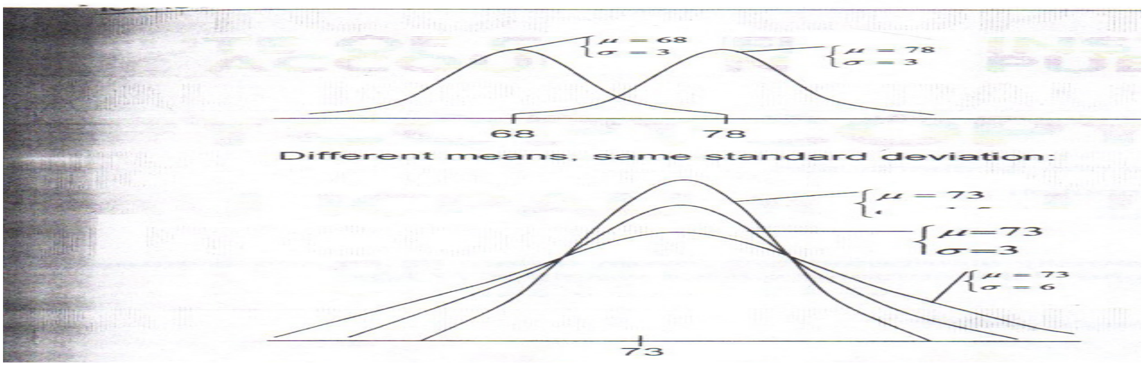
A normal distribution plays a pivotal role in statistical theory and practice, particularly in statistical inference and quality control.

a) Properties of a normal distribution curve

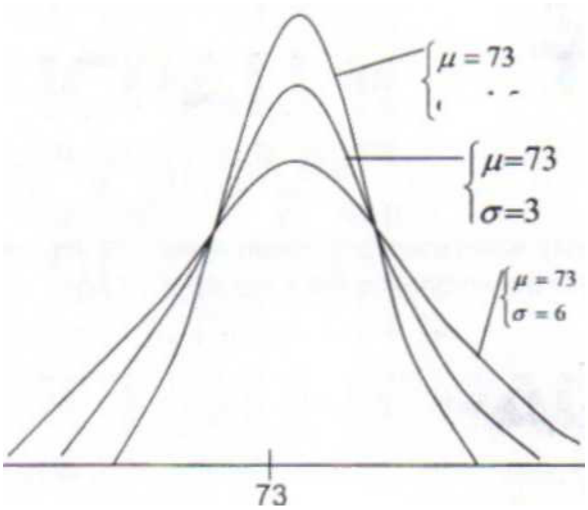
- All normal distribution curves are bell shaped.
- Symmetrical about the mean. This means the number of units in the data below the mean is the same as the number of units above the mean.
- The height of the normal curve is maximum at the value of the mean. This means that the normal distribution has the same value for the mean, median and the mode located at the centre of the distribution.
- Normal distribution is unimodal (has only one mode).
- The curve is continuous, for each value of x, there is a corresponding value of y
- The curve never touches the horizontal axis (asymptotic to the horizontal axis), that is, the height of the normal distribution curve declines as you move in either direction from the mean but never touches the base so that the tails of the curve on both sides extend indefinitely; and
- The total area under the normal curve is approximately equal to 1 (100%).

b) Importance of mean and standard deviation

When using a normal curve, there are two indispensable values: mean and standard deviation. Before a normal curve can be used to answer any question the mean and the standard deviation must be established. If a distribution is normal but not standard, we can convert the values to the standard normal distribution.



Different means, same standard deviation:



Same mean, different standard deviations.

Note:

- Changing the value of the mean while the standard deviation is constant shifts the distribution either to the right or left.
- On other hand changing the standard deviation while the mean remains constant creates kurtosis.
- The more the curve is spread the higher the standard deviation.

17.3. The Z-score

If X is a random variable that follows a normal distribution with mean p and standard deviation $\sigma\sigma$, is written as X ~N($\mu\mu$, $\sigma\sigma^2$). All variables that are normally distributed are
transformed into a standard normally distributed variable with mean 0 and standard deviation 1, are called Z scores. This process of transforming variables into Z values is called the standardization of the normally distributed variables.
A Z-sore is a value that indicates the number of standard deviations an observation (raw score) is above or below the mean of the distribution. The normally distributed variable X is converted to the Z-score using the formula:

$Z=\frac{(x-\mu)(x-\mu)}{\sigma \quad \sigma}$

where X = The value from a normal distribution
$\mu\mu$ = mean of the distribution

σ σ = standard deviation.

Example 17.1

The weight of students is normally distributed with mean weight 68 kg and a standard deviation of three. Okeny is one of the students whose weight is 77 kg.

Required:

Calculate the z score for Okeny's weight. Solution:

Using $Z = \frac{(x-\mu)(x-\mu)}{\sigma \quad \sigma}$

X=77 kg, $\mu\mu$ =68 and σ σ=3

$Z = \frac{77-6877-68}{3 \quad 3}$

$= \frac{99}{33} = 3$

Example 17. 2

Given that a random variable X follows a normal distribution with a mean of 500 and *a* Standard deviation of 100.

Required:

Find the z score for *a* value of 470.

Solution:

Z = , x=470, $\mu\mu$ =500 and (7=100 Z

$Z = \frac{470-500\,470-500}{100 \quad 100} = \frac{(-30)(-30)}{100 \quad 100} = -0.3$

17.4. Standard normal distribution tables

The standard normal distribution follows a normal distribution and has a mean 0 and standard deviation 1. Because 'the normal distribution curve is continuous it is only possible to find the probabilities (area) of values given in a range of values from the mean. The standard normal distribution table can be used to find area, probability or proportion (percentage) under normally distributed variables.

a) Finding area under the cumulative normal distribution

Example17. 3

Find the area under the normal curve in region z < 2.37.

Solution:

From the table, notice we need the area to the left (below) a z-score 2.37.

z	00	.01	.02	.03	.04	.05	.06	.07	.08	0.09
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893							0.4911		0.4916
2.4	0 4918							0 4932		0.4936

Below are few lines extracted from a normal distribution table:

Hence, area under the normal curve is such that (z<2.37) =0.5000+0.4911 = 0.9911.

b) Procedure for finding area under a normal curve

To find area under the cumulative normal distribution curve the following procedure is recommended:

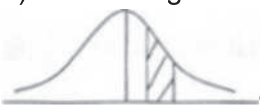
- Sketch of a normal curve.
- Indicate the boundaries and shade the desired area.
- The following shapes are important to locate the correct value.

i) Area between 0 and any value of Z:

a) on the right of the mean b) on the left of the mean

ii) In any tail

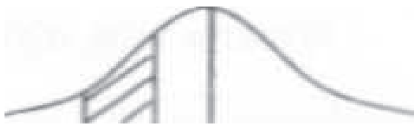
a) on the right tail b) on the left tail



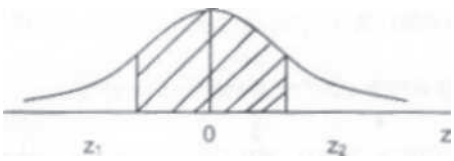
iii) Between two z - values on the same side of the mean

a) both values on the right of the mean

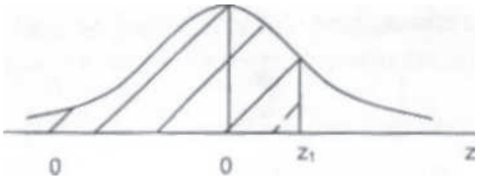
b) both values on the left of the mean



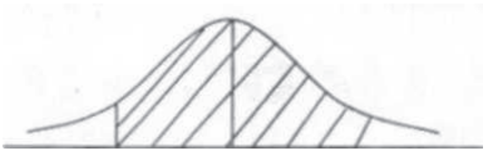
iv) Between two Z values on opposite sides of the mean



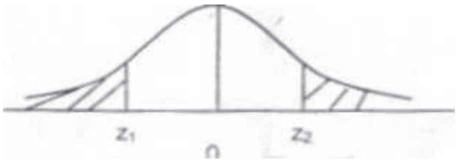
v) Less than any Z value to the right of the mean



vi) Greater than any z value to the to the left of the mean



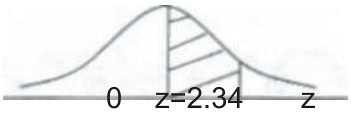
vii) In any of the two tails



Example 17.4

1. Find the area under the cumulative normal curve between 0 and any z value,

i) Consider z =0 and z= 2.34.

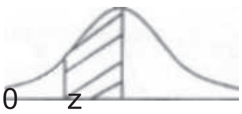


0 z=2.34 z

The table gives directly the value 0.4904.

ii) Consider z = 0 and z = -1.75.

The table does not give areas for negative values. But since the normal distribution is symmetrical about the mean, the area on the left of the mean, 0, is equal to the area on the right of the mean.



-1.75 0 z

Now get the value for +1.75 = 0.4599.

iii) Find the area to the right of 1.11.

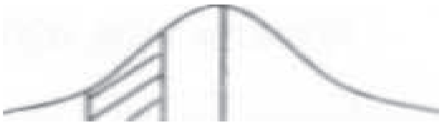
The required area is in the tail on the right of the curve.

0 1.11 z

Since the table gives the area between 0 and 1.11 directly. First find this area, then subtract this value from

0.5000 (half area). Area between 0 and 1.11 = 0.3665. Area to the right of 1.11 = 0.5000 -0.3665 = 0.1335.

iv) Find area to the left of - 1.93

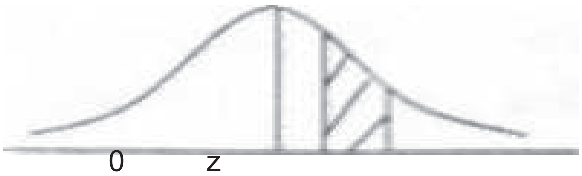


-1.93 z

The required area lies in the negative tail.

Its area is equal to the area in the positive tail
0.5000 - 0.4732 = 0.0268.

v) Find the area under the curve between any 2 z values on the right of 0.



0 z

Consider area between z = 2.0 and z= 2.47.
Since the two values are on the same right side of the mean.
Obtain area between 0 and 2.47 = 0.4932 and area between 0 and 2 = 0.4772.

; the difference between the 2 areas = 0.4932 -0.4772 = 0.0160.

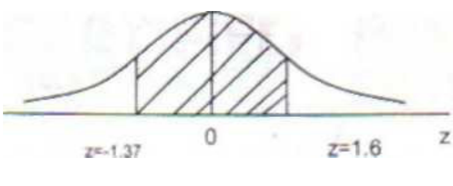
vi) Consider area between z = - 2.48 and -0.83.

The area between z = 0 and z= -2.48 is the same as area between 0 and 2.48 = 0.4934 and area between 0 and - 0.83 is the same as area between 0 and 0.83 = 0.2967.

0 z

The required area is the difference of the two areas = 0.4934 - 0.2967 = 0.1967.

vii) Find the area under the curve between any two Z values on opposite sides of the mean. Consider area between 1.68 and - 1.37.

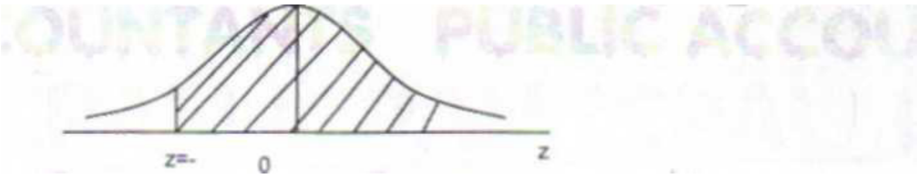


Since the two areas are on opposite sides of the mean 0. One must find both area and add them.
Area between 0 and 1.68 = 0.4535.

Area between 6 and - 1.37 = 0.4147.

Hence total area = 0.4535 + 0.4147 = 0.8682.

viii) Find the area to the right of Z=- 1.16.



The area between z = 0 and z = -1.16 = 0.3770,
Area to the right of 0 = 0.5000.
Total area = 0.3770 + 0.5000 = 0.8770.

In most, we shall require to find the probabilities of a Z-score in a defined range. A normal distribution curve can be used as a probability distribution curve for normally distributed values. The area under a cumulative normal distribution curve corresponds to probability. Therefore, problems involving probability are solved in the same way as finding area under a cumulative normal distribution curve. To find probabilities a special notation is used. For example, if the problem is to find the probability of any value Z between 0 and 2.32, this will be written as P(0 < Z <2.32).

17.5. Z- curves

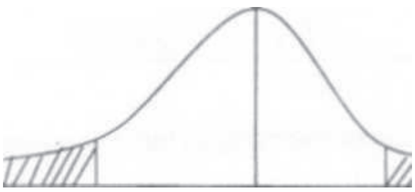
In general:

- P (z ≤z) = Area under the curve to the left of Z.
- For the probability in the interval (a b)

P (a ≤z ≤ b) = (Area to left of b) - (Area left of a)

Note: (z <0) = 0.5 (by symmetry)

Such that $P\ (Z \leq\ -z) = 1 - P(Z \leq z) = P(Z \geq z) P\ (Z \leq\ -z) = 1 - P(Z \leq z) = P(Z \geq z)$



-z z

P (Z ≤≤ -z) = 1- P (Z ≤≤ z) = P (Z ≥≥ z)

Example 17.5

Find:
i)P (Z ≤ -1.37)
ii)P(z ≥1.37)

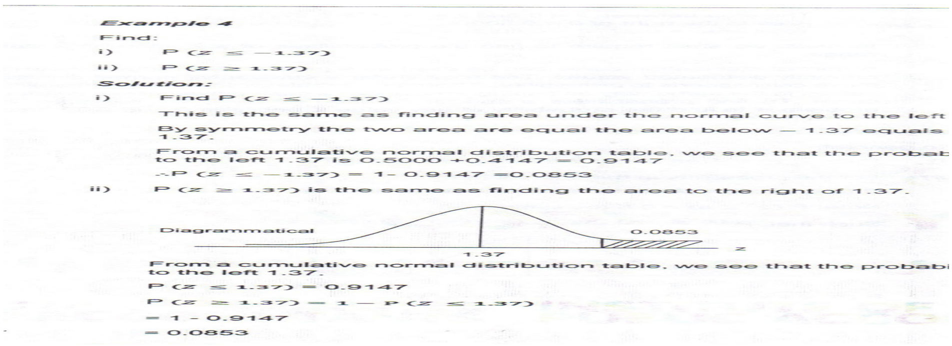
Solution:

i) Find P (z ≤-1.37)
This is the same as finding area under the normal curve to the left of- 1.37.

By symmetry the two area are equal the area below - 1.37 equals area above 1.37.
From a cumulative normal distribution table, we see that the probability or area to the left 1.37 is 0.5000 +0.4147 = 0.9147

P (z ≤≤-1.37) = 1- 0.9147 =0.0853

ii)P (z ≥≥1.37) is the same as finding the area to the right of 1.37.



Example 4

Find:

i) P (z ≤ −1.37)

ii) P (z ≥ 1.37)

Solution:

i) Find P (z ≤ −1.37)

This is the same as finding area under the normal curve to the left

By symmetry the two area are equal the area below — 1.37 equals 1.37.

From a cumulative normal distribution table, we see that the probab to the left 1.37 is 0.5000 +0.4147 = 0.9147

∴P (z ≤ −1.37) = 1- 0.9147 =0.0853

ii) P (z ≥ 1.37) is the same as finding the area to the right of 1.37.

From a cumulative normal distribution table, we see that the probabi to the left 1.37.

P (z ≤ 1.37) = 0.9147

P (z ≥ 1.37) = 1 − P (z ≤ 1.37)

= 1 - 0.9147

= 0.0853

From a cumulative normal distribution table, we see that the probability or area to the left 1.37

P(z ≤≤1.37) = 0.9147

P (Z ≥≥ 1.37) = 1 - P (Z ≤≤.1.37)

= 1 -0.9147

= 0.0853

Example 17.6

Calculate P (-o.155<**z <**1.60)

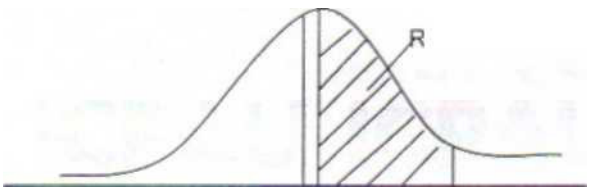
To calculate P (-0.155**'<z**< 1.60) is the same as finding area under the curve between two values on both the sides of the mean

We consider the area between and interpolate - 0.155 and 1.60.

P (z ≤≤**1.60)** is equal to area to the left of 1.60 =0.5000 +0.4552 = 0.9452

P (z ≥≥**-0.155)** = Area to left - 0.155 = area between 0 and 0.155 =0.0636.

So, P **(0.155 $<_z$< 1.60)** = 0.0636+0.4452= 0.4384

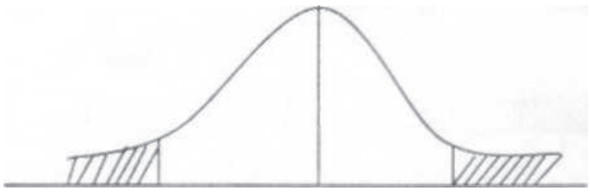


-0.155

Example 17.7

Find P (z <**-1.9** or z > **2.1)**

equired area



Solution: **Diagrammatically**

0

-1.9 2.1

These are two event (Z <-1.9) or (Z >2.1]

P **(Z <-1.9) or P (Z >2.1)**

P **(Z < -1.9) or P (Z >2.1)**

= 0.0287 + 0.0179

= 0.0466

How to obtain values of z given the probability

Example 17.8

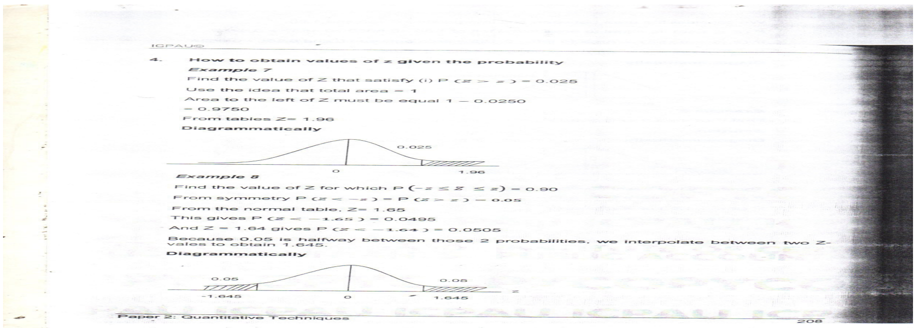
Find the value of Z that satisfy **(i)** P (z >**z**) = 0.025

Use the idea that total area = 1

Area to the left of Z must be equal 1 - 0.0250

= 0.9750

From tables Z= 1.96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4. **How to obtain values of z given the probability**

Example 7

Find the value of Z that satisfy (i) P $(z > z) = 0.025$

Use the idea that total area = 1

Area to the left of Z must be equal 1 – 0.0250

= 0.9750

From tables Z= 1.96

Diagrammatically

Example 8

Find the value of Z for which P $(-z \leq Z \leq z) = 0.90$

From symmetry P $(z < -z)$ = P $(z > z) = 0.05$

From the normal table, Z= 1.65

This gives P $(z < -1.65) = 0.0495$

And Z = 1.64 gives P $(z < -1.64) = 0.0505$

Because 0.05 is halfway between those 2 probabilities, we interpolate between two Z-vales to obtain 1.645.

Diagrammatically

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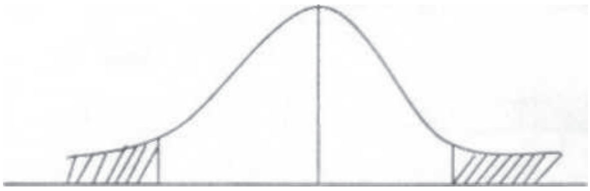
Example 17.9

Find the value of Z for which P (-z <**Z** < z) = 0.90. From symmetry P(z**<-z)** = P(z**>z)** = 0.

From the normal table, Z= 1.65. This gives P (z < -1.6S) = 0.0495

And Z = 1.64 gives P (z <-1.64) = 0.0505. Because 0.05 is halfway between those 2 probabilities, we interpolate between two Z-vales to obtain 1.645.



0.05 0.05

-1.645 0 1.645

Probability calculations concerning standard normal distribution

If X is a random variable that follows a normal distribution with mean $\mu\mu$ and standard deviation a, written as X ~N ($\mu\mu$, $\sigma\sigma^2$).

To solve problems by using the standard normal distribution, we assume that X is a random variable that follows a normal distribution with mean $\mu\mu$ n and standard deviation $\sigma\sigma$written as X ~N ($\mu\mu$, $\sigma\sigma^2$).Then transform the original variables X to standard normal distribution variables using the formula:
where, x = raw score
μ μ = mean of the distribution
$\sigma\sigma$ = standard deviation.

a) Procedure for calculating normal probabilities

Step 1: Sketch a normal distribution curve and indicate the required region by shading. Standardize the normal values to obtain the Z-score.

Step 2: Plan your solution according to the normal distribution table. Convert from X to Z by the formula Z = $\frac{x-\mu x-\mu}{\sigma \quad \sigma}$

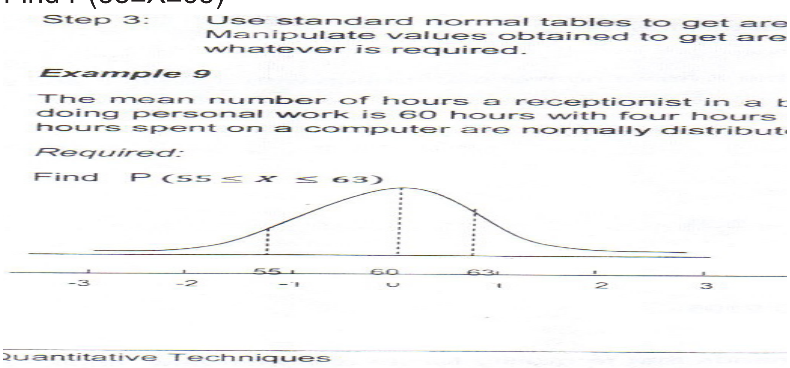
Step 3: Use standard normal tables to get area/probability. Manipulate values obtained to get area or probability or percentages whatever is required.

Example 17.**10**

The mean number of hours a receptionist in a busy office spends on a computer doing personal work is 60 hours with four hours standard, distribution. Assuming X hours spent on a computer are normally distributed.

Required:

Find P(55≤X≤63)


Step 3: Use standard normal tables to get are Manipulate values obtained to get are whatever is required.

Example 9

The mean number of hours a receptionist in a b doing personal work is 60 hours with four hours hours spent on a computer are normally distribut

Required:

Find P (55 ≤ X ≤ 63)

Quantitative Techniques

Here the standardized variable Z is given by.

$Z=\frac{x-60}{4}\frac{x-60}{4}$

X = 55 gives; Z = $Z=\frac{55-60}{4}\frac{55-60}{4}$
= -1.25

X - 63 gives: Z = $Z=\frac{63-60}{4}\frac{63-60}{4}$
= 0.75

P (55 <X < 63) = P (-1.25 <Z < 0.75)
= 0.7734-0.1056 = 0.6678
The necessary steps can be summarized as follows:

If X is a normally distributed N(u.a-), then P (a ≤≤x ≤≤b) =$\left[\frac{a-u}{\partial} \le z \le \frac{b-u}{\partial}\right]\left[\frac{a-u}{\partial} \le z \le \frac{b-u}{\partial}\right]$z < Z, where Z has the standard normal distribution.

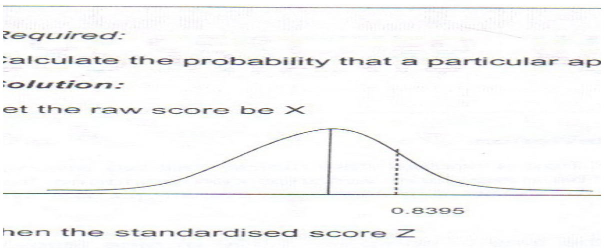
Example 17.**11**
Suppose a random variable X has a normal distribution with a mean 5 and standard deviation 2.
Required:
Determine P(1< X< 8). Solution:

Using the formula Z =$\frac{x-u}{\partial}\frac{x-u}{\partial}$

P(1<X<8)=p ($\frac{1-5}{2}\frac{1-5}{2}$< z < $\frac{8-8}{2}\frac{8-8}{2}$)
= P(-2<Z< 1.5)
= P(Z< 1.5) - P(Z< -2)
= 0.9332 - (1 -0.9773) = 0.9105
Example 17.**12**

A computer company demands that to qualify for an oral interview, applicants raw score in an aptitude test are normally distributed with a mean of 506 points and standard deviation of 81 points.

Required:
Calculate the probability that a particular applicant scored below 574 points.
Solution:
Let the raw score Then the standardized score Z



$Z = \frac{x-506}{81}\frac{x-506}{81}$

$Z = \frac{X-506}{81} = \frac{574-506}{81}\frac{X-506}{81} = \frac{574-506}{81}$ **=0.8395**
P (x<574) = P (Z < 0.8395) = 0.799

Question 17.13

In the Bible, the frequency distribution of the number of words per page is approximately normally distributed with a mean of 800 and with a variance of 2,500.

Required:

If three pages are selected at random what is the probability that none of them will have between 830 and 845 words?

Solution:

Let X be a random variable denoting the number of words per then X ~N (μ, σ²**).**

Where μ = 800 σ = 50

P(830 < X <845) = **P(**$\frac{830-800}{50} \leq z \leq \frac{845-800}{50}\frac{830-800}{50} \leq z \leq \frac{845-800}{50}$)
= P(- 0.6 < Z< 0.9)
=P(0<Z<0.6)+P(0<Z0.9)
=0.2257 + 0.3159
=0.5416

Finding the mean and standard deviation
Suppose that we know a variable X has a normal distribution, but we are not given the value for the mean and variance. The mean and variance can be found from any two probabilities concerning X and then formulate simultaneous equations to solve.
Example 17.14

In athletics, it generally taken that the time to complete 1,000 m race fallows a normal distribution. It was later found that the probability a competitor takes less than 160 seconds is 0.1 and that a competitor takes less than 165 seconds is 0.2
Required:

Determine the mean and standard deviation time for the race.
Solution:

Let X seconds be the time taken. X ~N (μ, σ).

Using the formula Z = Z = $\frac{x-\mu}{\sigma}\frac{x-\mu}{\sigma}$ Z = 0.1, 0.2
From normal distribution table Z_1= -1.282 and Z_2= - 0,842.

Then = Z = $\frac{160-\mu}{\sigma}\frac{160-\mu}{\sigma}$-1.282 and Z = $\frac{165-\mu}{\sigma}\frac{165-\mu}{\sigma}$=-0.842
160 — μ = -1.282 σand 165 - μ = -0.842 σ

Solving simultaneously μ = 174.6 sec. and σ = 11.4 sec
Self test questions

Question 17.1

Find the value of c in the following:
a) **P (o) < Z < c =** 0.2580.
b) P(c < Z <0) = 0.2580.
c) P (Z > c) = 0.0563.
Solution:

a) (0.70)
b) (-0.70)
c) (1.59)

Question 17.2
Suppose that a random variable Z has a standard normal distribution Find the following probabilities.
a) P (Z ≤ 1)
b) P (Z ≤ 4.7)
c) P (-1.87<Z<1)

Solution:
a) (0.8413)
b) (≈1)
c) (0.8106)

Question 17.3
If X is a random variable that has a normal distribution with a mean 50 and standard deviation 8
Required:
Find the probability that (≤**x** ≤39)
Solution:

(0.0776)
Question 17.4

A survey in a mining industry showed that workers had a mining experience of 17.34 years with a standard deviation of 11.34 years before they retire. Assuming workers form a normal distribution. Find the probability that a worker chosen at random retires:

a) When he/she retires has between 11 to 15 years of experience
b) When he/she has more than 15 years of experience.

Solution:
a) (0.1325)
b) (0.5832)

Question 17.5

A woodwork shop makes chairs and vanishes them. If the time for vanishing fallows a normal distribution with a mean time 55 minutes and standard deviation 4 minutes. The workshop closes at 5.00 pm every day. If an attendant starts to vanish a chair at 4.00 pm.
Required:
Determine the probability that he will finish vanishing before the workshop closes.
Solution: (0.8944)

Question 17.6

Given f(z) if a function for the normal curve.

Required:

Determine the area for the interval:

a) $z \geq \geq 2.35$.

b) $Z \geq \geq$ 2.35

Solution:

a) 0.4906

b) 0.0094.

Question 7

The estimated mean life of a family car is 8 years with a standard deviation of $1\frac{1}{4} 1\frac{1}{4}$ years. Assuming that the life of a new car is normally distributed.

Required:

Determine the probability that a family car older than 10 years is still be driven.

Solution: 0.0548

CHAP 18.

ESTIMATION AND CONFIDENCE INTERVAL

18.1. Study objectives

By the end of this chapter, you should be able to:

- Estimate the population mean from a large sample using the normal distribution;
- estimate the population mean from a small sample using the student's t distribution;
- use the contingency tables for the chi-square distribution; and
- estimate the population proportion from a large sample.

Definition of concepts

a) Estimation

Estimation refers to the process by which one makes inferences about a population, based on information obtained from a sample. It deals with the estimation of population characteristics (such as, the population mean and standard deviation) from sample characteristics (such as, the sample mean and standard deviation).

b) Point estimation

Involves using a statistic from a random sample to find an estimator for the corresponding population parameter. A point estimate of a population parameter is a single value used to estimate the population parameter. For example, the sample mean x is a point estimate of the population mean u. Note: Estimator is used for a Statistic; estimate is used for the numerical value of that statistic.

c) **Interval estimation**

Involves using the data from a random sample to find an interval within which an unknown population parameter is expected to lie with a given degree of confidence (probability). In statistics, interval estimation is the use of sample data to calculate an interval of possible (or probable) values of an unknown population parameter, in contrast to point estimation, which is a single number.

18.2. Standard error of mean

Standard error of mean also called standard deviation of mean is a method used to estimate the standard of a sampling distribution.

Standard error of the mean: $\delta_{\bar{x}}\delta_{\bar{x}}=\frac{\sigma}{\sqrt{n}}\frac{\sigma}{\sqrt{n}}$

a) Population mean

When the provided list represents a statistical population, then the mean is called the population mean. It is usually denoted by the letter "p." A population means may be defined as $\mu=\frac{\sum x}{n}$. Where; ∑x represents the sum of all the observations in the population; n represents the number of observations in the population,

b) Sample mean

When the provided list represents a statistical sample, then the mean is called the sample mean. The sample mean is denoted by (.t) It is a satisfactory estimate of the population mean for a sample.

18.3. Confidence interval (CI)

Statisticians use a confidence interval to describe the amount of uncertainty associated with a sample estimate of a population parameter.

a) How to interpret confidence intervals

Suppose that a 90% confidence interval states that the population mean is greater than 100 and less than 200. How would you interpret this statement?

Some people think this means there is a 90% chance that the population mean falls between 100 and 200. This is incorrect. Like any population parameter, the population mean is a constant, not a random variable. It does not change. the probability that a constant fall within any given range is always 0.00 or 1.00.

The **confidence level** describes the **uncertainty** associated with a **sampling method.** Suppose we used the same sampling method to select different samples and to compute a different interval estimate for each sample. Some interval estimates would include the true population parameter and some would not. **A** 90% confidence level means that we would expect 90% of the interval estimates to include the population parameter; A 95% confidence level means that 95% of the intervals would include the parameter; and so on.

b) Confidence interval data requirements

To express a confidence interval, you need three pieces of information;

- confidence level;
- statistic; and
- margin of error.

Given these inputs, the range of the confidence interval is defined by the sample statistic **±** margin of error. And the uncertainty associated with the confidence interval is specified by the confidence level.

c) Confidence interval

To obtain such a confidence interval, the first thing to determine is a **confidence level 1-$\alpha\alpha$**. This is a fixed probability that our estimate is within a certain range, which is chosen in advance. Oftentimes, the confidence level is 90%, 95%, 98%, or 99%. However, the higher the confidence level, the wider our range is, which could lead to some useless results. Therefore, a balance must be stroke between a high confidence level be stricken.

d) Margin of error

When estimating the population mean with the sample mean and when the population standard deviation a is known, the **maximum error of the estimate** for a given confidence level I - **a** also called the **margin of error for the mean is**

ME = $Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}}Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}}$

e) How to construct a confidence interval

There are four steps to constructing a confidence interval:

- Identify a sample statistic. Choose the statistic (e.g., sample mean, sample proportion) that you will use to estimate a population parameter.
- Select a confidence level. As we noted in the previous section, the confidence level describes the uncertainty of a sampling method. Often, researchers choose 90%, 95%, or 99% confidence

levels; but any percentage can be used.

- Find the margin of error. If you are working on a homework problem or a test question, the margin of error may be given. Often, however, you will need to compute the margin of error.

- Specify the confidence interval. The uncertainty is denoted by the confidence level. And the range of the confidence interval is defined by the following equation. Confidence interval = sample statistic ± margin of error, that is, once the margin of error has been determined, creating a confidence interval for the mean is easy. The lower bound of the interval is then $\bar{x}\bar{x}$— E while the upper bound of the interval is $\bar{x}\bar{x}$ + E

$\bar{x}$ $\bar{x}$ - $Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}}$ where $Z\frac{\sigma}{2}$ is the $\frac{1}{2}(100-y)\%$ if finding CI at y %

$Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}}$ where $Z\frac{\sigma}{2}$ is the $\frac{1}{2}(100-y)\%$ if finding CI at y %

$1-\sigma 1-\sigma$ is also called the degree of confidence, and it represents the probability that for a large random sample this interval will contain the population mean $\mu\mu$.

Example 18. 1

A random sample of 100 men from a population of 100,000 men was weighed to estimate the average weight of an adult male in Kigali. The average man in the sample weighs 18kgs, and the standard deviation of the sample is 3kgs

Required:

Find the 95% confidence interval.

Solution:

- Identify a sample statistic. Since we are trying to estimate the mean weight in the population, we choose the mean weight in our sample ($\bar{x}\bar{x}$ =18) as the

sample statistic.

- Find the margin of error, that is:
- find standard error. The standard error (SE) of the mean is:

SE= $\frac{\alpha}{\sqrt{n}}\frac{\alpha}{\sqrt{n}}$=$\frac{3}{\sqrt{100}}\frac{3}{\sqrt{100}}$=0.3

- find critical value. The critical value is a factor used to compute the margin of error. The critical value is a **z score.** From the tables, it is $\frac{z\alpha}{\sqrt{n}}\frac{z\alpha}{\sqrt{n}}$.=1.96

Compute margin of error (ME): $Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}}Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}}$=1.96x0.3=0.588

- Specify the confidence interval. The range of the confidence interval is defined by the **sample statistic ± margin of error.**

$\bar{x}\bar{x}$- $Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x}Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x}$+ $Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}}Z\frac{\alpha}{2}\frac{\sigma}{\sqrt{n}}$

= 180 - 0.588 <$\mu\mu$<180 + 0.588

- Therefore, this 95% confidence interval says that the population mean falls within the interval 180 + **0.588.**

Estimation of population mean

a) From large samples

The sample mean.$\bar{x}\bar{x}$ is used as the point estimator for the unknown population mean p. When the sample sizes are large n > 30, the sample means follow a normal or nearly a normal distribution with standard deviation, S_s also called the **standard error** of the mean given by$\bar{x}\frac{s}{\sqrt{n}}\bar{x}\frac{s}{\sqrt{n}}$, where S, the sample standard deviation is the estimator for **σ**, the population standard deviation.

At 1-σ level of significance the confidence Interval of the population mean, μ is given; μ = $\bar{x}\bar{x} \pm 2_{a/2}S\bar{x}\bar{x}$

Such that, **P (- $Z_{a/2}$<$\frac{\bar{x}-\mu}{s/\sqrt{n}}\frac{\bar{x}-\mu}{s/\sqrt{n}}$<$z_{a/2}$) = 1 - $\alpha\alpha$**

For 95% and 99% confidence levels, the most commonly used, $Z_{0^{\cdot}025}$ = 1.96 and $Z_{0,005}$ = 2.58.

Example 18.2

A juice making factory buys oranges from commercial fruit farmers strictly. In a bid to maintain high quality standards, a random sample of 750 oranges is picked and an investigation reveals that the mean weight is 358g with a standard deviation of 50g.

Required:

Calculate:

a) the standard error of the mean of the oranges bought.

b) the confidence interval of mean weight of the oranges at the level of confidence of:

i) 95%

ii) 99%

Solution:

$\bar{x}\bar{x}$= 358g n =750 S = 50g

a) Standard error of the mean

$=\frac{s}{\sqrt{n}}\frac{s}{\sqrt{n}}$

$=\frac{50}{\sqrt{750}}\frac{50}{\sqrt{750}}=\frac{10}{\sqrt{30}}\frac{10}{\sqrt{30}}$ = 1.826

b) i) At 95% level of confidence

$Z_{0.025}$= 1.96

Then, C.I = 358+- 1.96 (1.826) = 358+-3.S8

ii) At 99% level of confidence

$Z_{0.001}$ = 2.58

Then, C.I = 358 ± 2.58 (1.826) = 358 ±4.71

b) From small samples

For small samples, n < 30, the sample means, **x** do not follow a normal distribution. A student's t distribution is used for the estimation of the population mean, p.

The sample standard deviation S is used as an estimator for the population standard deviation. The confidence Interval for the population mean, u is given by;

$\mu\mu= \bar{x}\bar{x}=\pm ts\bar{x}\bar{x}$

Where the statistics t is given by $t=\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$

And S= $\sqrt{\frac{\sum(X-\overline{X})^2}{\sum n-1}}$ or $\sqrt{\frac{\sum f(X-\overline{X})^2\sum(X-\overline{X})^2}{\sum f-1 \quad \sum n-1}}$ or $\sqrt{\frac{\sum f(X-\overline{X})^2}{\sum f-1}}$

It should be noted that as the small sample sizes increases the t distribution tends to a normal distribution. The value of the t statistic can be obtained from the t distribution tables but it is necessary to obtain the degrees of freedom, given by v = n-1 where n is the sample size.

Example 18.3

"**M**" and Sons Ltd is an advertising company. A survey on the jobs that had been received to be done within a month was done on a random sample of eight jobs with an average of 60 words and standard deviation of five words.

Required:

Find the C.I for the mean number of words of the jobs to be done in the one month at:

a) 95% confidence level; and

b) 99% confidence level.

Solution

$\bar{x}\bar{x}=60, n=8, s=5, v=n-1=8-1=7$

a) $\mu = \bar{x}\mu = \bar{x} \pm ts\bar{x}\bar{x}$

but $t=2.365$, $s\bar{x}\bar{x}=\frac{s}{\sqrt{n}}\frac{s}{\sqrt{n}}=\frac{5}{\sqrt{8}}\frac{5}{\sqrt{8}}=1.7$

$\mu = 60 \pm 2.365x1.768$

$=60+-4.18$

b) $\mu = \bar{x}b)\ \ \mu = \bar{x} \pm ts\bar{x}\bar{x}$

But $t=3.499$, $s\bar{x}\bar{x}=\frac{s}{\sqrt{n}}\frac{s}{\sqrt{n}}=\frac{5}{\sqrt{8}}\frac{5}{\sqrt{8}}=\frac{2}{\sqrt{8}}\frac{2}{\sqrt{8}}=1.7$

$\mu = 60 \pm 3.499x1.768$

$=60 \pm 6.19$

18.4. Estimation of population proportions

When statistical information is given in form of proportions instead of the actual measures, the proportion represents an attribute of the population rather than the value of the variable.

Estimating population proportions from the sample proportions is similar to the procedure for estimating the population mean from the sample mean. However, the standard error of a sample proportion is given by the expression;

Standard error of a sample proportion = $\sqrt{\frac{PQ}{n}}\sqrt{\frac{PQ}{n}}$

Where; p= the population proportion

q= 1 - p

n= the sample size

Generally, for a random sample of size n (n>30), the proportion with a particular property, p; the **95%** confidence interval for the population proportion p is given by;

$(p-Z_{0.025})\sqrt{\frac{PQ}{n}}\sqrt{\frac{PQ}{n}}=, p+ Z_{0.025}\sqrt{\frac{PQ}{n}}\sqrt{\frac{PQ}{n}}), Z_{0.025=1.96}$

Usually written: $p\pm 1.96\sqrt{\frac{PQ}{n}}\sqrt{\frac{PQ}{n}}$

Other confidence intervals commonly considered are the **98%** and the **99%,** which are then given by:

99% confidence interval for p is p± **2.575** $\sqrt{\frac{PQ}{n}}$, Z0.02) = 2.575 and the $\sqrt{\frac{PQ}{n}}$, Z0.02) = 2.575 and the

98% confidence interval for p is p± **2.326** $\sqrt{\frac{PQ}{n}}$, Z0.01) $\sqrt{\frac{PQ}{n}}$, Z0.01)= 2.326

Example 18. 4

A soft drink depot operator in assessing the progress of the business decides to carry out a survey of the bottles that stayed beyond their expiry dates. Of a random sample of **200** bottles **30** had exceeded their expiry dates.

Required:

Calculate the:

95% confidence interval; and

99% confidence interval for the proportion of bottles in the depot that have exceeded their expiry date

Solution:

a) Given that p = $\frac{30}{200}\frac{30}{200}$=0.15

Then, q = 1 -0.15 = 0.85.

Also, n = 200.

Since **Z0.025** = 1 96, the 95% confidence interval = p±$Z_{0.025}$)$\sqrt{\frac{PQ}{n}}\sqrt{\frac{PQ}{n}}$

= 0.15 ±1.96$\sqrt{\frac{(0.15)(0.85)}{200}}\sqrt{\frac{(0.15)(0.85)}{200}}$

=0.15 ±0.0495

Thus, the 95% confidence interval is (0.1005, 0.1995)

b) Since $\mathbf{Z}_{0.005}$ = 2.575. The 99% confidence interval is given by:

P ± Zo.**005**$\sqrt{\frac{PQ}{n}}\sqrt{\frac{PQ}{n}}$

= 0.15 ±1.96$\sqrt{\frac{(0.15)(0.85)}{200}}\sqrt{\frac{(0.15)(0.85)}{200}}$

=0.15 ±0.0650

Example 18.5

In a bid to streamline their daily transactions, The United Peoples' Bank Ltd thoroughly scrutinised 600 entries made on a certain day of which 45 had errors. Find the proportion of all the entries that were made on that day which had errors, using a 98% level of confidence.

Solution:

Given, n = 600, P = $\frac{45}{600}\frac{45}{600}$ = 0.075, then q = 1 - 0.075 = 0.925. Also at 98% level of confidence, **Z0.01** = 2.326. The confidence interval is p ± 2.326 $\sqrt{\frac{PQ}{n}}\sqrt{\frac{PQ}{n}}$

=0.075 ± 2.326 $\sqrt{\frac{(0.075)(0.925)}{600}}\sqrt{\frac{(0.075)(0.925)}{600}}$

=0.075 ± 0.0250 = (0.050, 0.100)

Thus, the proportion of the accounts with errors on that day was between 5% and 10.

18.5. Self-test questions

Question 18.1

The weights of each of the ten specimens of **a** certain type of baking floor were found to be (in grams): 14.3, 13.8, 13.6, 14.6, 15.4, 14.8, 13.1, 14.2, 16.8, 15.1.

Given that the weights are approximately normally distributed with mean 14.57 and variance 1.44.
Required:
Construct a 95% confidence interval for p, the mean of the population weights.

Solution: 13.83<μμ< 15.31

Question 18.2
An investigation of the number of 260 annual sales of milk by **a** milkman employed in **a** particular dairy, yielded mean 15.3 and standard deviation 6.1. Find a) 98% and b) 95% confidence limits for the mean.
Solution:

a) 14.42<μμ< 16.18
b) 14.56<μμ<16.04

Question 18.3

Bags of Posho packed at Katwe Millers have masses which are normally distributed with mean 250g and a standard deviation of 15g.Determine the 95% confidence interval for the masses if a sample of 140 bags is checked.
Solution: 247.5<μμ< 252.49

Question 18.4
A survey carried out on the ages of 110 members who save with Bwavu SACCO. as the management is planning for future clients revealed that their ages were normally distributed and that 30% of the members were above 30 years while 52% were below 25 years Determine the 99% confidence interval of the population mean age y. of all the members in the catchment area of Bwavu SACCO.
Solution: 21.85<n < 27.0851

Question 18.5

An iPad is one of the sources of interruption at in a meeting reducing the level of attentiveness, a survey found that out of the 80 people in the meeting, 58 said they were interrupted five or more times an hour by the iPad messages. Find the 95% confidence level of the proportion of members who are interrupted five or more times an hour.

Solution: 0.6272<μμ<0.8228

Question 18. 6

Mr. Mukisa is the head of the quality control department in a mushroom packaging factory. He randomly selected 15 packets among the output in a given season and discovered upon his survey that the mean weight was 60g and standard deviation of 9g.

Required:

Determine confidence interval of the mean weight of the entire output in this season at 95% level of significance.

Solution: 55.908< μ < 60.4092 at 95% level of significance.

Question 18.7

A random sample comprises of 620 torches. The technicians who sorted them out registered mean defects of eight and variance of 12 per torch.

Required:

Determine the confidence interval for the mean defects in all the torches at 99% level of significance.

Solution: $7.641 < \mu < 8.359$

Question 18.8

A hospital administrator wishes to determine the appropriate sample size to use to study the daily patients they receive in the facility. Operating at 1% of the true proportion

Required:

Determine the sample size the administration should recommend

Solution n= 100

CHAP 19

HYPOTHESIS TESTING

19.1. Study objectives

By the end this chapter, you should be able to:

- define basic concepts used in hypothesis testing;
- formulate null and alternative hypotheses for application involving a single population
- mean or proportion;
- define Type I and Type II errors:
- compute the probability of a Type II error;
- list the steps used in testing hypothesis;
- formulate a decision rule for testing a hypothesis; test the statistic at a given critical value; and
- Determine critical values (acceptance or rejection regions).

19.2. Hypothesis testing

Hypothesis testing is part of inferential statistics that uses data samples to make conclusions on population parameters. The purpose of this type of inference is to determine whether a certain claim or assumption or hypothesis about a parameter can be justified using statistical evidence. The purpose of hypothesis testing is also to make valid decisions about population parameters based on analysis of the sample.

What is a hypothesis?

Every day in business, whether the business is small or large, profit or non-profit making; managers are faced with decisions. For example, a quality controller must examine products on the basis of a sample taken from the product. In order to make such statistical decisions we make claims about population parameters to be tested. These claims are known as hypotheses.

A hypothesis is a stated claim (assumption) about a population parameter, such as population mean or population proportion. Sample data is then used to check the reasonableness of the claim.

- Rwanda 's mean monthly rainfall is 150 mm.
- The proportion of adults in Kigali city with cell phones is 0.68.
- The mean height of army recruits is 1.8 m.

To justify these claims are true or false we carry out hypothesis testing.

19.3. Types of hypothesis

There are two types of hypothesis: The null hypothesis: This is a hypothesis to be tasted. We use the symbol (Ho) to represent the null Hypothesis. H_0 is a statement about a value of a population parameter. The alternative hypothesis: This is a hypothesis to be considered as alternative to the null hypothesis. We use the symbol (H_a).to represent alternative hypothesis. H_a is the statement that is accepted if the sample data provide enough evidence that Ho is false. The alternative hypothesis represents population values other than those contained in the null hypothesis.

19.4. Hypothesis formulation

The objective of a hypothesis is to use sample information to decide whether to accept or to reject the null hypothesis about 3 population value The only two hypothesis testing decision would be reject Ho or do not reject Ho.

a) The null hypothesis is a test concerning a population mean μ should always specify a single value for the parameter. This means that all null hypothesis should always be of the form $\mu = \mu o \mu = \mu o$ where $\mu o \mu o$ is a specified value. We can therefore express the null hypothesis as $Ho: \mu = \mu o$ $Ho: \mu = \mu o$

b) Alternative hypothesis. The choice of alternative hypothesis depends on and should reflect the purpose of performing the hypothesis test.

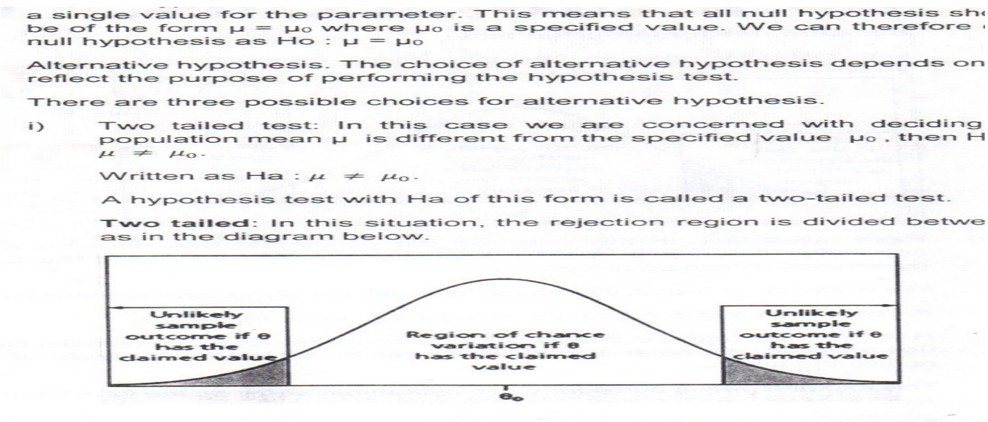
There are three possible choices for alternative hypothesis.

i) Two tailed test: In this case we are concerned with deciding whether a population mean μ is different from the specified value μo, then Ha should be

Written as Ha:**μ#μ0μ#μ0**

A hypothesis test with Ha of this form is called a two-tailed test.

Two tailed: In this situation, the rejection region is divided between two tails as in the diagram below.


a single value for the parameter. This means that all null hypothesis sh
be of the form $\mu = \mu_o$ where μ_o is a specified value. We can therefore
null hypothesis as $Ho : \mu = \mu_o$

Alternative hypothesis. The choice of alternative hypothesis depends on
reflect the purpose of performing the hypothesis test.

There are three possible choices for alternative hypothesis.

i) Two tailed test: In this case we are concerned with deciding
population mean μ is different from the specified value μ_o, then H
$\mu \neq \mu_o$.

Written as Ha : $\mu \neq \mu_o$.

A hypothesis test with Ha of this form is called a two-tailed test.

Two tailed: In this situation, the rejection region is divided betwe
as in the diagram below.

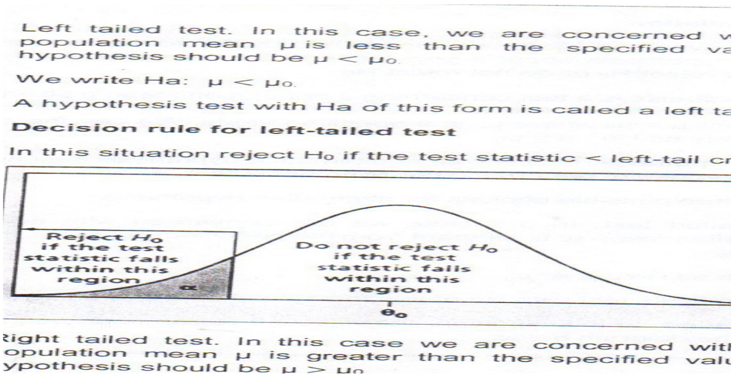
ii) Left tailed test. In this case, we are concerned with deciding whether a population means p is less than the specified value po. The alternative hypothesis should be $\mu < \mu <$uo

We write Ha: μ<μo

A hypothesis test with Ha of this form is called a left tailed test.

Decision rule for left-tailed test

In this situation reject Ho if the test statistic$<$ left – tail critical value $<$ left – tail critical value



Left tailed test. In this case, we are concerned w
population mean μ is less than the specified va
hypothesis should be $\mu < \mu_0$

We write Ha: $\mu < \mu_0$

A hypothesis test with Ha of this form is called a left ta

Decision rule for left-tailed test

In this situation reject H_0 if the test statistic < left-tail cr

Right tailed test. In this case we are concerned wit
opulation mean μ is greater than the specified valu
ypothesis should be $\mu > \mu_0$

iii) Right tailed test. In this case we are concerned with deciding whether a population mean p is greater than the specified value $\mu o \mu o$. The alternative hypothesis should be $\mu > \mu o\, \mu > \mu o$

We write Ha: $\mu > \mu o \mu > \mu o$ A hypothesis test with Ha of this form is called a right test.

A hypothesis test is called one tailed if it is left or right tailed.

However, although the definition for null and alternative hypotheses given here uses a single parameter, this definition can be extended to include other forms of distributions.

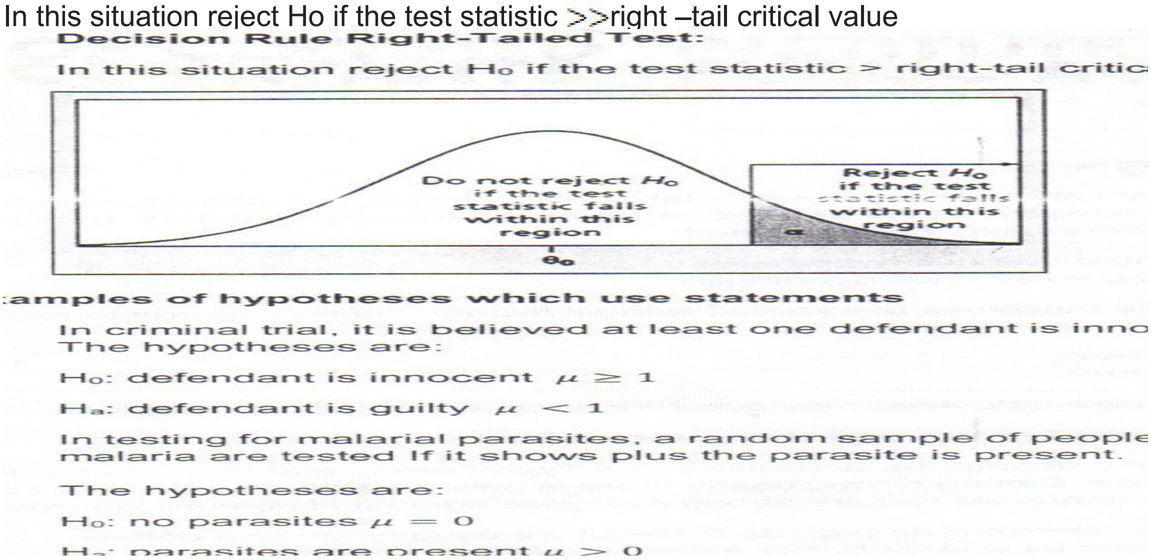
This explains why a compound null hypothesis can take the form Ho $\mu.\leq\ or\ \geq\ \mu 0 \mu.\leq\ or\ \geq\ \mu 0$

To state the hypotheses correctly the claim from the statements should be translated into mathematical symbols. The basic symbols used are as follows:

i) Equal t o =	ii) Less than <
iii) Not equal #	iv) greater than >
v) less or equal ≤	vi) greater or equal ≥≥

Decision rule Right –tailed test:

In this situation reject Ho if the test statistic >>right –tail critical value



Decision Rule Right-Tailed Test:

In this situation reject H_0 if the test statistic > right-tail critic

amples of hypotheses which use statements

In criminal trial, it is believed at least one defendant is inno
The hypotheses are:

H_0: defendant is innocent $\mu \geq 1$

H_a: defendant is guilty $\mu < 1$

In testing for malarial parasites, a random sample of people
malaria are tested If it shows plus the parasite is present.

The hypotheses are:

H_0: no parasites $\mu = 0$

H_a: parasites are present $\mu > 0$

Examples of hypotheses which use statements

1. In criminal trial, it is believed at least one defendant is innocent The hypotheses are:

   Ho: defendant is innocent $\mu \geq 1 \mu \geq 1$

   H_a: defendant is guilty $\mu < 1 \mu < 1$

2. In testing for malarial parasites, a random sample of people suspected to have malaria are tested

If it shows plus the parasite is present.

The hypotheses are:

Ho: no parasites $\mu\mu = 0$

H_a: parasites are present $\mu > 1 \mu > 1$

3. A researcher feels that playing soft music during exam will affect results of the exam. The researcher is not sure whether the grades will be higher or lower. In the past, the mean score was 72.

Ho: $\mu\mu = 72$

H_a: μ μ #72

4. The average weight loss of a sample who exercise 30 minutes per day for six weeks is 1.8 kg.

H_0: $\mu\mu$ = 1.8 k,q

H_a: $\mu\mu$ 1 8 kg

19.5. Types of errors

Careful thought should be given in establishing the null and alternative hypotheses, since the conclusion reached may depend on the hypothesis being tested. In making a decision two types of errors may be committed:

- Rejecting a true null hypothesis.
- Accepting a false null hypothesis.

Three possible outcomes associated with all hypothesis testing problems include:

- No error.
- Type I error.
- Type II error.

Only one of these outcomes may occur for each test of a null hypothesis.

Type I and Type II errors and their probabilities

Let the symbol a denote the probability of a Type I error, and let p denote the probability of a Type II error. Setting allow means there is only a small chance of rejecting Ho when it is true. This means we require strong evidence against Ho before we reject it.

We sometimes choose an as high as 0.10 but we usually choose between 0.05 and 0.01. A frequent choice for a is 0.05. For a fixed sample size, the lower we set a, the higher is 3, and the higher we set a, the lower is p.

Example 19.1

Juice making companies have been recently under pressure from consumer groups which claim that bottles prices have been increasing while bottles are filled with less than the advertised amount of 330 milliliters. A manager responsible for making sure that the juice filling machines operate correctly, samples bottles of juice every one hour and based on the sample results decides whether to adjust the machines or not. The results revealed the mean amount to be 315ml and standard deviation of 15ml from a sample of 64 bottles.

There are two possible states of nature:

i) If the bottles are filled with 330ml or more of juice on average (the machines are operating correctly).

ii) If the bottles are filled with less than 330ml on average (the machines are not operating correctly).
The manager must base his or her decision about the filling process on the results of the hourly sample. When a decision is based on sample results, sampling error must be expected.
Type I error: Rejecting the null hypothesis when it is true. This occurs with probability α.

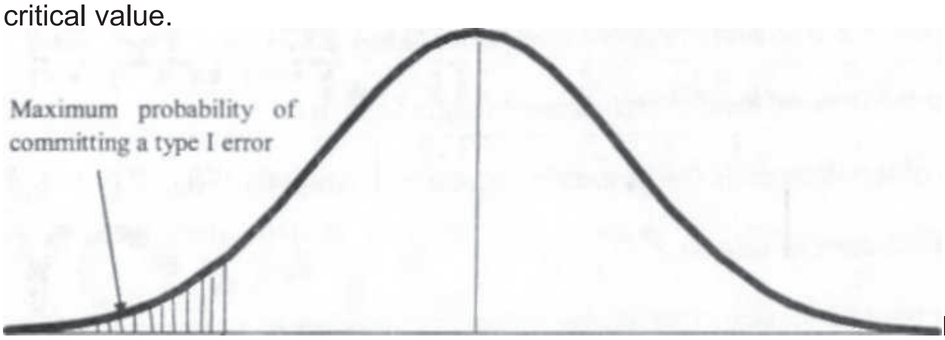
Type II error: Failure to reject the null hypothesis when it is false; this occurs with probability β.

The table below shows decision rule and types of errors:

Decision	If H_o is True	If H_o is False
Reject H_0	Type 1 error (a risk)	Correct decision
Don' t reject **Ho**	Correct decision	Type it error(B risk)

Critical value and rejection region

The job of the decision maker is to establish a cut-off point that can be used to separate sample results that should lead to rejecting Ho from sample results that will lead to accepting Ho. This cut-off point is called a critical value.



In the example, the hypotheses are:

<-------------------------- ► u=300

Reject Ho   Accept Ho

H_0: $\mu \geq \mu \geq$330mls
H_a: $\mu \leq \mu \leq$ 330mls
and if the level of significance a = 0.10; for a=0.1, the critical value is $Z_{0.1}$= $\mathbf{-Z_{0.1}}$ = -1.28. So, the decision

rule is: Reject Ho if Z computed < -1.28
Otherwise do not reject Ho

The sample standard error, s* =15, n =64and sample mean x =315

$$Z_{computer} = \frac{\bar{x}-u_r \bar{x}-u_r}{\frac{sr}{\sqrt{n}} \quad \frac{sr}{\sqrt{n}}} = \frac{315-330\ 315-330}{\frac{15}{\sqrt{64}} \quad \frac{15}{\sqrt{64}}} = -8$$

Since $-8 < Z_{01} -8 < Z_{01}$= -1.28 (i.e., lies in the rejection region), reject Ho. Thus, the bottles are filled with less than 300ml on average (machines are not operating correctly).

19.6. Procedure for testing hypothesis

i) State the null (H_0) and the alternative hypothesis (H_a).
ii) Select the desired level of significance (alpha level, a).
Level of significance is the probability of reject H_0 when it is true.

iii) **Choose a sample size n.**

Select the test statistic. This is the value from a sample data used to determine whether to reject H_0. There are many test statistics. In this chapter, we shall only consider z-standard normal, t - students and x^2 chi-square. In hypothesis testing

the mean is given as (p) and the standard deviation given as a.If the value of o is known or sample size is large, the test statistic of z-value is computed from the formula z = .$\frac{x-u}{\alpha}\frac{x-u}{\alpha}$

iv) **Formulate the decision rule.**

A decision rule is a statement of a specific condition under which H_0 is rejected and condition under which it is not rejected. The region or area of rejection defines the location of all those values that are so large or small that the probability of their occurrence under a true H_0 is small.

v) **Make a decision.**

In this step, you compare the test statistic with the critical value and make a decision to reject H_0 based on sample information.

The procedure for hypothesis testing can be summarised in four steps:
Step 1: State the null hypothesis H_0 and alternative hypothesis Ha.
Step 2: Decide on appropriate sample statistic and test statistic. The selection of a test statistic is based on:

i) Ho
ii) the chosen sample statistic.
iii) the tenable assumptions concerning the population distributions.

Assumptions underlying the sampling distribution of **z**,**t**and **X** ^{2}**.**

Step 3: Decide on the level of significance **a** and the sample size n. $\alpha\alpha$and n together with the sampling distribution test statistic under H_0 determine the region for rejecting Ho.

The location and size of the region for rejection of H_0 are determined by Ha and **a** respectively. An experimenter attempts to select level of significance so that the rejection region contains values of the test statistic that have a low probability of occurrence of Ho being true but a high probability if Ha is true.

Step 4: Obtain the sample statistic and compute the test statistic. If the value of the test statistic falls in the region of rejection, H_0 is rejected in favour of Ha. If the test statistic falls outside the region of rejection, the researcher may either accept Ho or suspend making a decision concerning it.

19.7. Hypothesis testing using single means (z test for mean)

Example 19.2

A lighting company has decided to build a test sample of 1,000 light bulbs that are assumed to be a random sample before it begins full scale production. The sample results showed the mean of 704 hours and a standard deviation of 150 hours.

Required:

Determine at 5% level of significance whether the mean life of the new light bulbs exceeds the old bulb average of 700 hours.

Hypotheses: H_0: $\mu_x \leq\leq 700$ hrs
H_a: $\boldsymbol{\mu_x} \geq\geq 700$ hrs
$\alpha\alpha = 0.05$

For αα = 0.05. in the right-tailed test, the critical value is $z_{0.05}$ = 1 -65.

So the decision rule is: reject Ho if $Z_{calculated}$ >1.65,otherwise do not reject Ho
Test statistics:

$z = \frac{\bar{x}-u}{\frac{\alpha}{\sqrt{n}}} \frac{\bar{x}-u}{\frac{\alpha}{\sqrt{n}}} = \frac{704700}{\frac{150}{\sqrt{1000}}} \frac{704700}{\frac{150}{\sqrt{1000}}} = 0.8433$

Since 0.8433 < 1.65, we accept the Ho. Thus, the mean life of the new light bulbs does not exceed that of the old bulbs.

Example 19. 3

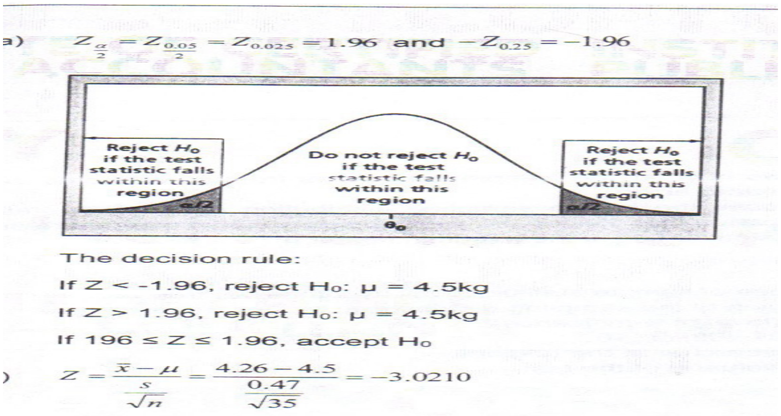
A fish factory that processes and packs fish in 4.5kg packets. Consumers claim that the packets may not have the correct weights. A quality control analyst randomly selects 35 packets from the population of packets processed on a particular day. He calculates the mean and standard deviation as x =4.26 kg and s_x = 0.47kg. The analyst wants to test consumers claim at 5% level of significance.
Required:

a) Formulate a decision rule.
b) Test Ho: u=4.5 kg against H_a: u # 4.5 kg at 5% level of significance.

Solution:

The null hypothesis should be refuted for the value of x either lower than u=4.5kg or above u=4.5kg. The situation illustrates a two-tailed hypothesis test.
Hypotheses: H_0: μ = 4.5 kg
H_a: μ # 4.5 kg
α =0.05

a) $Z\alpha\alpha_{/2} = Z\frac{0.05}{2} = 1.96 \text{and} - Z0.25 = -1.96\frac{0.05}{2} = 1.96 \text{and} - Z0.25 = -1.96$


a) $Z_{\frac{\alpha}{2}} = Z_{\frac{0.05}{2}} = Z_{0.025} = 1.96$ and $-Z_{0.25} = -1.96$

The decision rule:
If Z < -1.96, reject Ho: μ = 4.5kg
If Z > 1.96, reject Ho: μ = 4.5kg
If 196 ≤ Z ≤ 1.96, accept Ho

$Z = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} = \frac{4.26-4.5}{\frac{0.47}{\sqrt{35}}} = -3.0210$

The decision rule:

If Z < -1.96, reject H_0: μ = 4.5kg
If Z > 1.96, reject H_0: μ = 4.5kg
If 196 S Z < 1.96, accept H_0

b) $Z = \frac{\bar{x}-\mu}{\frac{\alpha}{\sqrt{n}}} = \frac{4.26-4.5}{\frac{0.47}{\sqrt{35}}} \frac{\bar{x}-\mu}{\frac{\alpha}{\sqrt{n}}} = \frac{4.26-4.5}{\frac{0.47}{\sqrt{35}}}$=-3.0210

Because Z = -3.0210 < -1.96, reject H_0: u = 45 in favor of H_a:u# 4.5kg at 5% level of significance.

Alternatively, Z = 3.0210 >Z **0025** > 1.96. Thus, the mean weight of fish differs from 4.5kg.

Summary table for hypothesis testing:

The null and alternative hypotheses must be stated together and the null hypothesis contains the equal sign as shown

Left-Sided Test	Two-Sided Test	Right-Sided Test
H_0- $\theta \geq \theta\theta \geq \theta o$	H_0: $\theta = \theta\theta = \theta o$	H_0- $\theta \leq \theta\theta \leq \theta o$
H_1: $\theta < \theta\, \theta < \theta o$	H1θ#θθ#θo	H_1: $\theta > \theta\, \theta > \theta o$

Decision rule

- A test statistic shows how far the sample estimate is from its expected value, in terms of its own standard error.
- The decision rule uses the known sampling distribution of the test statistic to establish the critical value that divides the sampling distribution into two regions.
- Reject Ho if the test statistic lies in the rejection region.

8. Level of significance, α

Defines unlikely values of sample statistic if null hypothesis is true. Defines rejection region of the sampling distribution. Is designated by a. (level of significance). Typical values are .01. .05, or .10. Is selected by the researcher at the beginning. Provides the critical value(s) of the test.

Example 19.4

The quality controller in an industry claims that on average the number of calls on equipment sold by CBA is more than 15 per week. To investigate this claim, 36 records were randomly checked, with result $\bar{x}\bar{x}$=17 and s^2 =9 for the sample data.

Required:

Does the sample evidence contradict the quality controller's claim at 5% level of significance?

The parameter of interest is $\mu\mu$= 15 number of calls on the equipment sold by CBA. H_0: $\mu \leq \mu \leq$ 15 against H_a>15.

Using z as a test statistic Z=$\frac{\bar{x}-u\bar{x}-u}{\alpha/n\alpha/n}$ We reject H_0 if for Z > $z_{0.025}$ = 1.645

Substituting s for o we calculate Z=$\frac{17-1517-15}{3/36\ \ 3/36}$=4

The calculated z = 4 and at a =0.05 Za = 1.645. this implies Z > $Z_{0.05}$.

Since 4 > 1.645, there is sufficient evidence to reject H_0.

Distinction between one tailed and two tailed tests of significance.

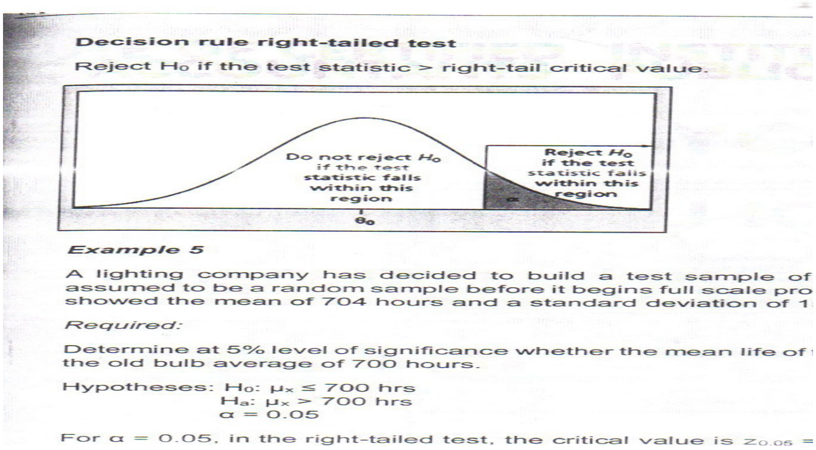
One tailed: One tailed hypothesis test will assume one of the two forms, depending on the way the null and the alternative hypotheses are stated. Examples of these two forms are:

H0: $\mu_x \leq\leq$ 60 marks

H_a: μ_x> 60 marks

Decision rule right-tailed test

Reject Ho if the test statistic > right-tail critical value.



Decision rule right-tailed test

Reject H_0 if the test statistic > right-tail critical value.

Example 5

A lighting company has decided to build a test sample of assumed to be a random sample before it begins full scale pro showed the mean of 704 hours and a standard deviation of 1

Required:

Determine at 5% level of significance whether the mean life of the old bulb average of 700 hours.

Hypotheses: H_0: $\mu_x \leq 700$ hrs
H_a: $\mu_x > 700$ hrs
$\alpha = 0.05$

For $\alpha = 0.05$, in the right-tailed test, the critical value is $z_{0.05}$ =

Example 19.5

A lighting company has decided to build a test sample of 1,000 light bulbs that are assumed to be a random sample before it begins full scale production. The sample results showed the mean of 704 hours and a standard deviation of 150 hours.

Required:

Determine at 5% level of significance whether the mean life of the new light bulbs exceeds the old bulb average of 700 hours.

Hypotheses: Ho: μ < 700 hrs
H_a: μx > 700 hrs
α = 0.05
For α= 0.05, in the right-tailed test, the critical value is $z_{0.05}$ = 1-65. So the decision rule is:
reject Ho if calculated >1.65, otherwise do not reject Ho Test statistics:

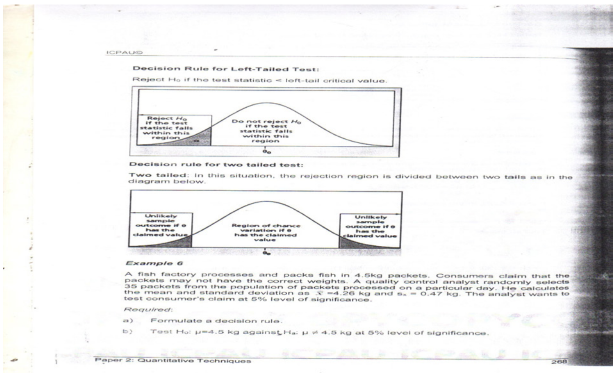
$$z = \frac{x-\mu}{\frac{\delta}{\sqrt{n}}} = \frac{704-700}{150/\sqrt{1000}} z = \frac{x-\mu}{\frac{\delta}{\sqrt{n}}} = \frac{704-700}{150/\sqrt{1000}} = 0.8433$$

Since 0.8433 < 1.65, we accept the Ho. Thus, the mean life of the new light bulbs does not exceed that of the old bulbs.

Ho: **μx** ≤≤ 18 years
H_a: $\mu_x <$ 18 years

Decision rule for left tailed test: two tailed test:

reject Ho if the test statistc $< left - tail\ critical\ value < left - tail\ critical\ value$



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Decision Rule for Left-Tailed Test:

Reject H_0 if the test statistic < left-tail critical value.

Decision rule for two tailed test:

Two tailed: In this situation, the rejection region is divided between two tails as in the diagram below.

Example 6

A fish factory processes and packs fish in 4.5kg packets. Consumers claim that the packets may not have the correct weights. A quality control analyst randomly selects 35 packets from the population of packets processed on a particular day. He calculates the mean and standard deviation as $\bar{x}$ =4.26 kg and s_x = 0.47 kg. The analyst wants to test consumer's claim at 5% level of significance.

Required:

a) Formulate a decision rule.

b) Test H_0: μ=4.5 kg against H_a: μ ≠ 4.5 kg at 5% level of significance.

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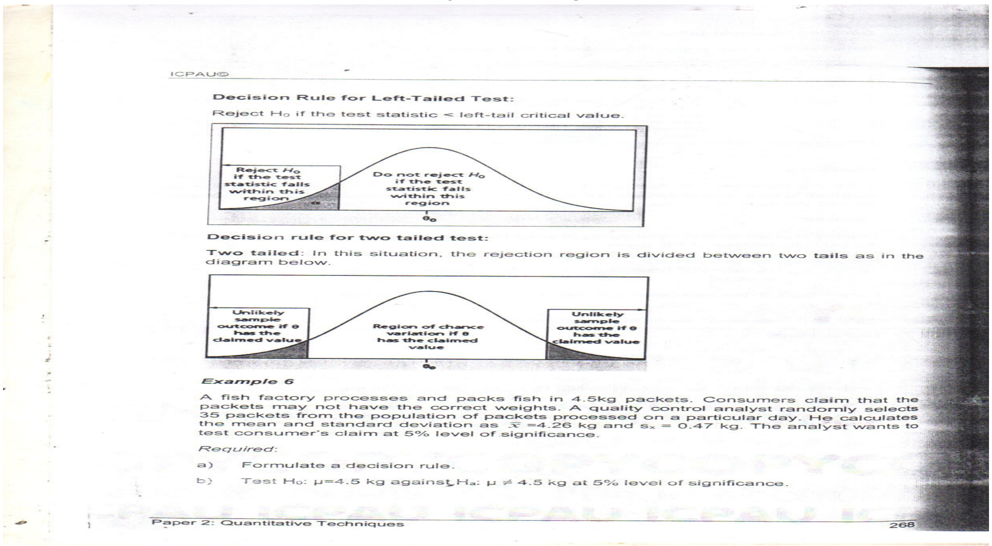
Decision Rule for two tailed test:

Two tailed: In this situation, the rejection region is divided between two tails as in the diagram below



ICPAU

Decision Rule for Left-Tailed Test:

Reject H_0 if the test statistic < left-tail critical value.

Decision rule for two tailed test:

Two tailed: In this situation, the rejection region is divided between two tails as in the diagram below.

Example 6

A fish factory processes and packs fish in 4.5kg packets. Consumers claim that the packets may not have the correct weights. A quality control analyst randomly selects 35 packets from the population of packets processed on a particular day. He calculates the mean and standard deviation as $\bar{x}$ =4.26 kg and s_x = 0.47 kg. The analyst wants to test consumer's claim at 5% level of significance.

Required:

a) Formulate a decision rule.

b) Test H_0: μ=4.5 kg against H_a: μ ≠ 4.5 kg at 5% level of significance.

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Example 19.6

A fish factory processes and packs fish in 4.5kg packets. Consumers claim that the packets may not have the correct weights. A quality control analyst randomly selects 35 packets from the population of packets processed on a particular day. He calculates the mean and standard deviation as **x** =4.26 kg and s_x = 0.47 kg. The analyst wants to test consumer's claim at 5% level of significance.

Required:

a) Formulate a decision rule.

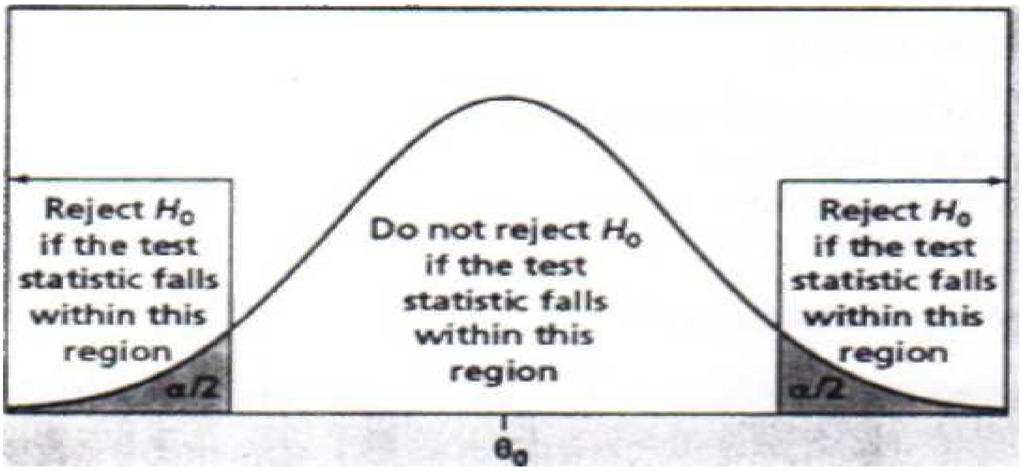
b) Test H_0: μ=4.5 kg against H_a: μ# 4.5 kg at 5% level of significance.

Solution:

The null hypothesis should be refuted for the value of $\overline{xx}$either lower than μ=4.5kg or above μ=4.5kg. The situation illustrates a two-tailed hypothesis test.

Hypotheses: Ho: μ = 4.5 kg
H_a: μ # 4.5 kg
α =0.05

a) $Z_{a/2}=Z_{0.05/2}=Z_{0.025}$ = 1.96and $-Z_{0.25}$=-1.96



The decision rule:

If Z < -1.96, reject H_0: u = 4.5kg

If Z > 1.96, reject H_0: u = 4.5kg

If 196 ≤≤Z ≤≤ 1.96, accept H_0

b) $Z = \frac{\bar{x}-u}{\frac{\alpha}{\sqrt{n}}}\frac{\bar{x}-u}{\frac{\alpha}{\sqrt{n}}} = Z = \frac{4.26-4.5}{\frac{0.47}{\sqrt{85}}}\frac{4.26-4.5}{\frac{0.47}{\sqrt{85}}} = -3.0210$

Because Z = -3.0210 < -1.96, reject H_0:μ = 45 in favour of H_a:μ #4.5kg at 5% level of significance. Alternatively, Z = 3.0210 >Z0.025 > 1.96. Thus, the mean weight of fish differs from 4.5kg.

Summary table for hypothesis testing:

Left-Sided Test	Two-Sided Test	Right-Sided Test
H_0- θ ≥ θθ ≥ θo	H_0: θ = θθ = θo	H_0- θ ≥ θθ ≥ θo
H_1: θ ≤ θ θ ≤ θo	H1θ#θθ#θo	H_1: θ ≤ θ θ ≤ θo

Statistical Hypothesis Testing

- The direction of the test is indicated by H_a:
- > indicates a right-tailed test
- < indicates a left-tailed test
- # indicates a two-tailed test

When to use a one- or two-sided test:

- A two-sided hypothesis test (i.e., q t qo) is used when direction (< or >) is of no interest to the decision maker.
- A one-sided hypothesis test is used:

i) when the consequences of rejecting Ho are asymmetric; or

ii) Where one tail of the distribution is of special importance to the researcher.

- Rejection in a two-sided test guarantees rejection in a one-sided test, other things being equal.

9. Decision rule

- A test statistic shows how far the sample estimate is from its expected value, in terms of its own standard error.

- The decision rule uses the known sampling distribution of the test statistic to establish the critical value that divides the sampling distribution into two regions.
- Reject Ho if the test statistic lies in the rejection region.

Level of significance, α

- Defines unlikely values of sample statistic if null hypothesis is true.
- Defines rejection region of the sampling distribution.
- Is designated by a, (level of significance).
- Typical values are .01, .05, or .10.
- Is selected by the researcher at the beginning.
- Provides the critical value(s) of the test.

Example 19.7

The quality controller in an industry claims that on average the number of calls on equipment sold by CBA is more than 15 per week. To investigate this claim, 36 records were randomly checked, with result $\overline{xx}$=17 and s^2 =9 for the sample data.

Required:

Does the sample evidence contradict the quality controller's claim at 5% level of significance?

The parameter of interest is μ = 15 number of calls on the equipment sold by CBA.

H_0: u ≤≤ 15 against H_a>>15.

Using z as a test statistic Z = Z=$\frac{\bar{x}-u\bar{x}-u}{\sigma/n\sigma/n}$We reject H_0 if for Z > $z_{0.05}$ = 1.645

Substituting for s for a we calculate Z = Z=$\frac{17-15}{3/36} = 4\frac{17-15}{3/36} = 4$

The calculated z = 4 and at a =0.05 Za = 1.645. this implies Z > $z_{0.25}$.

Since 4 > 1.645, there is sufficient evidence to reject H_0.

11. Hypothesis testing between two means

To perform such a test, two conditions are necessary:

- The samples must be independent of each other (i.e., sample one from the first population is not related to the sample two selected from the second population).
- The sample size of each sample must be greater than 30 or each sample must be drawn from a population with a known standard deviation.

19.8. We use the Z-test to test the difference between two population means M1 and M2.

If these requirements are met, then the sampling distribution x, -x, (the difference of the sample means) is a normal distribution with mean and standard error of

$$\mu_{x_1-x_2} = \mu_1 - \mu_2 \ and \ \delta_{x_1-x_2} = \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Example19. 8

A civil welfare activist organisation claims that there is a big difference in the mean weekly wages earned by workers in government departments and in private sector. The results of a

Private sector	Government department
x , = Frw 60,900	**x ,** = Frw 64,300
si= Frw 12.000	s_2 = Frw 15,000
n1=100	n2=100

Random surveys of 100 workers from each sector were as follows:

Test the civil welfare organization's claim at 5% level of significance.

Solution:

The claim is that there is a big difference in the mean weekly wage for the workers.

So, Ho: u1 = u_2

H_a: u1 #**p2**

The test is a two -tailed test and at the level of significance is $\alpha\alpha$=0.05, the critical values are -1.96 and 1.96. The rejection regions are Z < -1.96 and Z > 1.96.

Both samples are large, si and **S2** we calculate the Z value. S2 we calculate the value

$$\delta_{\bar{x}_1}\delta_{\bar{x}_1}\text{-}\bar{x}_2\bar{x}_2=\sqrt{\frac{s1^2}{n1}}+\frac{s2^2}{n2}\sqrt{\frac{s1^2}{n1}}+\frac{s2^2}{n2}=\sqrt{\frac{(12000)^2}{100}}+\frac{(15000)^2}{100})=1921\sqrt{\frac{(12000)^2}{100}}+\frac{(15000)^2}{100})=1921$$

Using the Z-test the standardized test statistic is

$$Z=\frac{(\bar{X}-\bar{X}2)-(\bar{U}1-U2)(\bar{X}-\bar{X}2)-(\bar{U}1-U2)}{\delta_{\bar{x}_1}-\bar{x}_2 \qquad \delta_{\bar{x}_1}-\bar{x}_2}=\frac{(60900-64300)-0(60900-64300)-0}{1921 \qquad 1921}=-1.770$$

$-1.77 \in 1-\alpha\ acceptance\ area$

The calculated Z is not in the rejection region. Therefore, we fail to reject the null hypothesis. At 5% level of significance. We conclude that there is not enough evidence to reject the claim.

Example 19.9

A study was conducted to compare the length of time it took men and women to perform an assembly task in a busy company. Independent random samples of 50 men and 50 women were employed each was timed in minutes on identical tasks. The results are summarized in the table below.

	Men	women
N	50	50
$\bar{x}$	42	38
s2	18	14

Required:

Do the data provide sufficient evidence to suggest that there is a difference between completion time for men and women in this task at a= 0.05 level of significance?

Solution:

Let$\bar{x}\bar{x}_1$ and $\bar{x}\bar{x}_2$ be mean completion time for men and women, respectively. We are interested in the difference.

$H_{0:}$ $\bar{x}\bar{x}_1$- $\bar{x}\bar{x}_2$ = 0

$H_{a:}$$\bar{x}\bar{x}_1$- $\bar{x}\bar{x}_2$ #o

Because samples are large we use Z test statistic given by the formula:

$$Z=\frac{\bar{x}1-\bar{x}2 \quad \bar{x}1-\bar{x}2}{\alpha(\bar{x}1-\bar{x}2)\alpha(\bar{x}1-\bar{x}2)}=\frac{\bar{x}1-\bar{x}2 \quad \bar{x}1-\bar{x}2}{\sqrt{\frac{s^2{}_1}{n_1}+\frac{s^2{}_2}{n_2}}\sqrt{\frac{s^2{}_1}{n_1}+\frac{s^2{}_2}{n_2}}}$$

We use two tailed test $z_{0.25\ =1.96\ \text{and because}}$ $z > z0.25 > z0.25$

$$\frac{42-3842-38}{\sqrt{\frac{18}{50}+\frac{14}{50}}\sqrt{\frac{18}{50}+\frac{14}{50}}}=5$$

The calculated Z value =5 >Z_α =1.96. Hence, reject H_0 and conclude that there is a significant difference in the completion times of men and women.

Example 19.**10**

A researcher claims that urban boys are fatter than village boys. To test the claim 50 boys aged 18 from town and boys aged 18 from rural areas were selected at random and their weights were taken and the results were recorded as follows:

	Urban	Rural
Mean weight(kg)	65.2	64.5
Standard deviation(kg)	2.5	2.8

Required:

Test the hypothesis that urban boys are more fat than village boys at a = 0.05 level of significance.

Solution:

This requires A Z-test since the samples are large and it is one tailed test.

H_0: μ1>μ2

H_a: μ1< μ_2

Then the Z = $\frac{66.2-64.566.2-64.5}{\sqrt{\frac{2.5^2}{50}+\frac{2.8^2}{50}}\sqrt{\frac{2.5^2}{50}+\frac{2.8^2}{50}}}$ =1.319

The critical value of Z at a = 0.05 level of significance for one tailed test = 1.64. Since the calculated Z value 1.319 < zα = 1.64.

We accept H_0 and conclude that urban boys are fatter than village boys.

19.9. Testing hypothesis for small samples

Student's t-test is a method used for testing hypotheses about the mean of small samples drawn from a normally distributed population. It is particularly used when the population standard deviation is not known. The sample is usually considered small when the sample size is less than 30.

When the population standard deviation is unknown, the z-test becomes inappropriate for testing hypothesizes involving means. A different test called student's t-test is instead

Used.

a) Properties of students' t-test

The t-test is similar to the standard normal distribution in the following ways:

i) It is continuous, bell shaped and is symmetrical about the mean.

ii) The mean, median and mode are equal and are located at the centre of the distribution.

iii) The curve is an asymptotic to the horizontal axis (x-axis).

The t-distribution differs from the normal distribution in the following ways:

i) The variance for t is greater than one.

ii) The t- distribution is a family of curves based on degrees of freedom.

Each time the degree of freedom changes a new distribution is created which is a value related to the sample size df = (n-1).

iii) As the sample size (degrees of freedom) increases, the shape of the t-distribution approximates the standard normal distribution.

iv) The t distribution is more flat or more spread out than the standard normal curve.

b) To test a hypothesis using t distribution

Use the formula for t = $\frac{\bar{x}1-\mu\bar{x}1-\mu}{s\sqrt{n}\ s\sqrt{n}}$ with n - 1 degrees of freedom where:
$\overline{x}\overline{x}$ is the mean of the sample μ μ is the hypothesized population mean s- is the standard deviation of the sample n is the number of observations in the sample (sample size)

Note: Since the population standard deviation o is unknown, the sample standard deviation s is used instead. When you test the hypothesis by using t-test, you follow the same procedure for Z-test already explained, except you use t formula and t-distribution tables.

Example 19.11

Find the critical value of t for $\alpha\alpha$ = 0.05 with d.f = 16 for a right tailed test from the table.

Solution:

Find the 0.05 column in the top row and df= 16 in the left hand column. The two intersect at the appropriate critical value 1.746 as shown in the following table.

Portion of the t distribution table.

Df Y	Probability						
	0.10 0.20	0.05 0.10	0.025 0.05	0.01 0.02	0.005 0.010	0.0025 0.0050	Q 2Q
14	1.345	1.761					
15	1.341	1.753					
16	1.337	1.746					
17	1.333	1.740					
18	1.330	1.734					
19	1.328	1.729					

Example 19.**12**

Find the critical t values for α α = o.o1 with degree of freedom = 22 for a left tailed test.
Solution:

Find the 0.01 column in the row labelled one tail and df =22 in the left column. The critical value is -2.508 since the test is one tailed.

Example 19.**13**
Find the critical values for $\alpha\alpha$= 0.10 with d.f = 18 for a 2 tailed.
Solution:

Find the column labelled 0.10 in the row labeled two tailed and df = 18 row. The critical values are 1.734 and -1.734.

Example 19.**14**
Bank XYZ claims that loan department reports that the mean cost to process a bank loan from commercial bank is $60. A bank comparison showed this amount was too larger than most other banks, so they instituted a cost cutting measure. To evaluate the effect of cost cutting measures, Bank XYZ selected a random sample of 26 recent loan applications. The mean cost per claim was $57 and the standard deviation was $10.

Required:
Can the XYZ conclude that the cost cutting measures were effective at 0.01 level of significance.
Solution:
Let use the general steps highlighted for Z-distribution.
Ho $\mu \geq \mu \geq$ $ 60.
H_a< $ 60
Level of significance 0.01. df = 26 -1 = 25, to = 2.485.

Test the statistic by $t=\frac{\bar{x}-u\bar{x}-u}{s\sqrt{n}\ s\sqrt{n}}$

In this problem $\overline{xx}$=57 mean of the sample. uu=60 the hypothesized population mean.
s= 10 the standard deviation of the sample n = 26 the number of observations in the sample

- $t=\frac{57-60\,57-60}{10\sqrt{26}\,10\sqrt{26}}$=1.530

Since -1.530 lies in the region to the right of the critical value -2.485.
Accept Ho at 0.01 level of significance. There is no statistically significant difference between % and n . This indicates that the cost cutting measures have not reduced
the mean cost per claim to less than $60. The difference of $3 could be by chance.
Example 19.15
A human resource manager claims that the mean starting salary for a receptionist is Frw 24,000. A sample of 10 receptionists' salaries has a mean of Frw 23,450 and a standard deviation of Frw 400

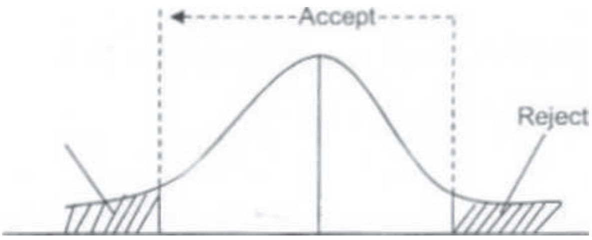
Required:
Is there good evidence from the sample to reject the human resource manager's claim at $\alpha\alpha$ = 0.05?
Solution:

H_0: μ = Frw 24.000 (claim) and H_a: μ# Frw 24,000.
Here we use 2-tailed test d.f =10-1 = 9 for $\alpha\ \alpha$ = 0.05, critical values for t = 2.262 and -7.662.

Test the Statistic: $t=\frac{\bar{x}-u\bar{x}-u}{s\sqrt{n}\,s\sqrt{n}}$

$\frac{23.450-24.00\,23.450-24.00}{400\sqrt{10}\qquad 400\sqrt{10}}$ =-4.35

Comparing -4.35 and -2.262 as shown in the figure:



Reject Ho, since -4.35 < -2.262. as shown in the figure

There is good evidence to reject human resource manager's claim that the starting salary for receptionists is Frw 24,000.

Example 19.**16**

An inspector of schools claims that on average, part time teachers in rural schools earn less than Frw 60 per hour. A random sample of eight districts was selected and the hourly pay was recorded as follows:

60, 56, 60, 55, 75, 55, 60, 55

Required:

Is there good evidence to accept the Inspector of Schools claim at 10% level of Significance.

Solution:

Ho $\mu \geq 60$ **and Ha** $\mu < 60$ $claim\mu \geq 60$ **and Ha** $\mu < 60$ $claim$

At

at $\alpha\alpha$= 0.10 d.f = 8 - 1 = 7

The critical value for tables -1.415. First find mean and standard deviation.

$$\bar{x} = \frac{60 + 56 + 60 + 55 + 75 + 55 + 60 + 55}{8}$$

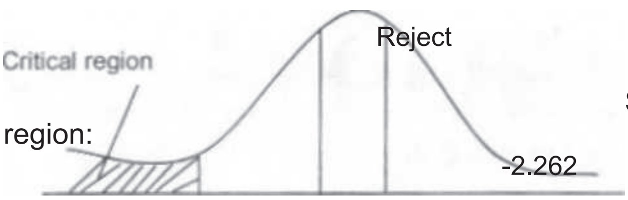
=58.88sh per hour

Sd $(\alpha)s = 5.08\alpha)s = 5.08$

T=$\frac{\bar{x}-u\,\bar{x}-u}{s\sqrt{n1}s\sqrt{n1}}$ =$\frac{58.88-6058.88-60}{5.08\sqrt{8}\quad 5.08\sqrt{8}}$

= 0.624

Compare -0.624 and -1.415



Since, -0.62 > -1.415, accept H_0. -0.624 lies in non-critical region:

2.262

There is no good enough evidence to support inspector's claim that the mean earning per hour is less than Frw 60.

Difference between the means from small samples

Testing the difference between two means from a normal distribution follows three basic assumptions:

-1.415 -0.624

- The sampled populations must follow a normal distribution.
- The two samples are from independent populations.
- The standard deviations of the two samples are equal.

Consider samples of sizes m and n2 with standard deviations Si and S2, chosen from a normal population. Assume $\bar{x}\bar{x}_1$ and $\bar{x}\bar{x}_2$ are the means of the samples.

Then to test the hypothesis that the two samples were obtained from the same populations (i.e., µ1 = µ2and S1 = S2).

We use $t = \frac{\bar{x}1 - \bar{x}2}{\sigma\sqrt{\frac{1}{n1}+1/n2}\sigma} \frac{\bar{x}1 - \bar{x}2}{\sqrt{\frac{1}{n1}+1/n2}}$ **and s=** $\sqrt{\frac{n1xs^2 + n2xs^2}{n1+n2+2}} \sqrt{\frac{n1xs^2 + n2xs^2}{n1+n2+2}}$

With d.f = $\mathbf{n_1+n_2-2}$

$\bar{x}1x1$ is the mean of the first sample.

$\bar{x}2\,\bar{x}2$ is the sample mean of second sample.

S is the standard deviation of the sample.

$\mathbf{n_1}$ is the sample size of first sample.

$\mathbf{n_2}$ is the sample size of second sample.

Example 19.17

The knowledge level of 16 students from TEAM Business college showed a mean of 107 with a standard deviation of 10, while the knowledge level of 14 students from MAT college showed a mean of 112 and standard deviation of 8.

Required:

Is there a significant difference between the student knowledge levels of the two colleges at 0.01 level of significance?

Solution

Let µ µ and $\mu 2$ be the population means of the two colleges.

H_0: µ1µ1 = µ2µ2. that is, there is no difference between the means of the two colleges

H_a: µ1µ1#µ2µ2

Test statistic under H_0

$\frac{\bar{x}-\bar{x}}{\sqrt{\frac{1}{n1}+\frac{1}{n2}}} \frac{\bar{x}-\bar{x}}{\sqrt{\frac{1}{n1}+\frac{1}{n2}}}$ **where S=** $\sqrt{\frac{n1xs^2 + n2xs^2}{n1+n2+2}} \sqrt{\frac{n1xs^2 + n2xs^2}{n1+n2+2}}$

$S = \sqrt{\frac{16x10^2 + 14x8^2}{16+14-2}} \sqrt{\frac{16x10^2 + 14x8^2}{16+14-2}}$ =9.5469

$T = \frac{112-107}{9.5469\sqrt{\frac{1}{16}+\frac{1}{14}}} \frac{112-107}{9.5469\sqrt{\frac{1}{16}+\frac{1}{14}}}$ **=1.431**

d.f = 16 + 14 - 2 = 28 at αα= 0.01 On the basis of two tailed test at a = 0.01 t = to.oos = 2.76 since t < t α2 α2, 1 431 < 2.76

We accept H_0at 0.0 level of significance. Conclude that there is no significant difference between the knowledge levels of students in the two colleges.

Example 19.18

It is desired to establish if there is a significant difference between the average amount of pocket money given to male and female students joining university. A random sample of eight male and ten female students were selected and the amount of pocket money was established. From each sample group the mean and standard deviation were calculated and the results were:

Male student	Female student
n1= 8	n_2=10
$\overline{xx}$ =20.05	X_2=17.00
S1=2.00	s_2=1.50

All money in 100.000 Rwandan Francs .

Required:

Test if there is any significant difference in the average amount of pocket money c .'en to students at 5% level of significance.

Solution:

H_o: there is no significant difference between the pocket money.

H_0:.u1 = μ2

H_a: u1 # μ2

Where t= $\frac{\bar{x}-\bar{x}}{\sqrt{sp2(\frac{1}{n1}+\frac{1}{n2})}}\frac{\bar{x}-\bar{x}}{\sqrt{sp2(\frac{1}{n1}+\frac{1}{n2})}}$ **and**

and sp^2 $=\frac{(n1-1)s1^2+(n1-1)s2^2(n1-1)s1^2+(n1-1)s2^2}{n1+n2-2 \quad n1+n2-2}$

sp^2 $=\frac{(8-1)2^2+(10-1)1.5\ ^2(8-1)2^2+(10-1)1.5\ ^2}{8+10-2 \quad 8+10-2}$

$t=\frac{20.5-17.00}{\sqrt{3.01(\frac{1}{8}+\frac{1}{10})}}\frac{20.5-17.00}{\sqrt{3.01(\frac{1}{8}+\frac{1}{10})}}$**=4.25**

The critical value at a = 0.05, from a two-tailed test and df = 8+10-2 = 16 t = 2.12. This implies calculated **t** > t a = **4.25 > 2.12.** Then reject H_0.

Example 19.19

Five RODA new drivers were given a test to determine their knowledge of the Kigali streets. The test results were not considered to be satisfactory. The five drivers were recommended for a two-day workshop training on the various streets in Kigali. The following table represents the scores obtained by the drivers before and after the workshop training.

Driver number	Score before training	Score after training
1	8	10
2	9	12
3	12	15
4	15	19
5	12	15

Required:

At 0.05 level of significance, test whether the drivers have improved knowledge of Kigali streets.

Solution:

H_0: let the mean deviation be < 0 H_a: let the mean deviation be > 0. Since the sample is small we use a t- test

$T=\frac{\bar{d}}{sd/\sqrt{n}}\frac{\bar{d}}{sd/\sqrt{n}}$

Before	After	D	d	
8	10	2	3	1

9	12	3	3	0
12	15	3	3	0
15	19	4	3	1
12	15	3	3	0
Total		**15**		**2**

Mean deviation $\frac{1515}{3\ 3}$**=3**

Standard deviation $(sd) = \sqrt{\frac{\Sigma(d-\bar{x})^2}{n-1}} = \sqrt{\frac{2}{4}}(sd) = \sqrt{\frac{\Sigma(d-\bar{x})^2}{n-1}} = \sqrt{\frac{2}{4}}$=0.7071

$$t = \frac{\bar{d}}{\frac{sd}{\sqrt{n}}} = \frac{\bar{3}}{\frac{0.7071}{\sqrt{7}}} = 9.487$$

the critical value of t at a = 0.05, df = 4 is 2.353. Since the calculation t = 9.487 is greater, the to = 2.353, then, H_0 is rejected.

19.10. The chi-square ($\chi^2\chi^2$) test of hypothesis

The chi-square (x^2) statistic is a non-parametric statistical technique used to determine if a distribution of observed frequencies differs from expected frequencies.

Chi-square unlike Z, and t which use mean and variances: data used in chi-square has to satisfy the following conditions:

- Data must be randomly drawn from a population, be recorded in rows and column of frequency samples which are independent.
- The observed frequencies must be large and data values of independent and dependent variables must be mutually exclusive.

Properties of chi-square

- The chi-square is never negative this is because the differences between (f_0) and expected (f_e) is squared, that is $_{,(fo\text{-}fe)}{}^{2.}$
- There is different chi-square distribution for each degree of freedom. The degree of freedom is determined by k - 1; where k is the number of categories. The shape of chi-square does not depend on the size of the sample, but on the number of categories used.
- The chi-square distribution is positively skewed, but if the degrees of freedom become large the distribution approximates the normal.

Types of chi-square test

There are two types of chi-square test:

Chi-square test of goodness of fit

This test compares the expected and observed values to determine how well the sample predicts the population.

Goodness of fit means how well a statistical model fits a set of observations in the sample.

A measure of goodness of fit summarises the discrepancies between observed values and expected values in a distribution.

Chi-square test of independence

This compares two sets of data to determine whether the two groups are distributed differently among the categories.

Chi-square for independence is used to determine the relationship between two variables of a sample. In this contest independence means that the two have factors which are not related.

a) **Formula for the computation of the chi-square (x^2)**

$\chi^2 \chi^2 = \sum \frac{(O-E)^2}{E} \sum \frac{(O-E)^2}{E}$ where O is the observed frequency of any value, E is the expected frequency of the value.

The test **is** concluded by comparing the computed value and the values from **the** Chi-square tables for a given significance level and a number of degrees of freedom

using extract of X^2 distribution **table below**

Degrees of freedom (v)	**Level of significance**	
0	**5%**	**1%**
1	**3.841**	**6.635**
2	**5.991**	**9.210**
3	**7.815**	**11.345**
* * * *	* * * *	* * * *
24	**36.415**	**42.979**
25	**37.652**	**44.314**

b) Contingency table

In order to determine the association between two sets of measurements, the correlation coefficient between sample pairs is appropriate. However, this method is inadequate for testing the association between two attributes since these are non - measurable characteristics. The x^2test is appropriate where by the observed frequencies with which items in a sample do or do not possess both attributes are compared with the expected frequencies estimated on assumption that the two attributes are independent. A sample item is classified with respect to an attribute into at least two mutually exclusive and collectively exhaustive classes. Where the number of classes is just two, the items possessing the attribute fall into one class and those not possessing it fall into the other. When the sample items are classified with respect to a second attribute, also into two classes, then each item falls into one of the four possible classes according to whether or not it possesses both the first and second attributes as illustrated in the table below

		Attribute P		
		P	P'	Total
Attribute Q	Q	W	X	w+x
	Q"	Y	z	y+z
	Total	w+y	x+z	n

A sample of size n is therefore classified into four mutually exclusive classes PnQ, PnQ, P'nQ & P∩∩Q', the observed frequency for each class being w, y, x, z respectively as shown in the table.

$P(PnQ) = P(P).P(Q) = \frac{(w+y)(w+x)}{n^2}\frac{(w+y)(w+x)}{n^2}$

And expected frequency of items in the top left hand cell of the table

$$n\left\{\frac{(w+y)(w+x)}{n^2}\right\}$$

$$\frac{(w+y)(w+x)}{n^2}$$

Similarly, the other expected frequencies can be generated.

c) Computation of the x^2 value for the distribution

Considering the distribution in the table above and taking e, $e_2 e_3$ and e_4 as the expected frequencies in each cell with corresponding observed frequencies w, x, y, z, respectively.

$$x^2 = \frac{(w-e1)^2 (w-e1)^2}{e1 \quad e1} + \frac{(w-e2)^2 (w-e2)^2}{e2 \quad e2} + \frac{(w-e3)^2 (w-e3)^2}{e3 \quad e3} + \frac{(w-e4)^2 (w-e4)^2}{e4 \quad e4}$$

where $e_1 = \frac{(w+y)(w+x)(w+y)(w+x)}{n \quad n}$, $e2 = \frac{(w+z)(w+x)(w+z)(w+x)}{n \quad n}$, $e3 = \frac{(w+y)(y+x)}{n}$, $e4 = \frac{(w+z)(w+x)}{n}$ $= \frac{(w+y)(y+x)}{n}$, $e4 = \frac{(w+z)(w+x)}{n}$

The table above is referred to as a contingency table and as the classification is into two classes with respect to each attribute it is a 2 x 2 contingency table.

To approximate the x^2 value for a mxn contingency table from the x^1 distribution tables, the degrees of freedom,

v =(m-1)(n-1)

In the above case:

v = (2-1)(2-1)

v = 1 degree of freedom

d) Procedure for chi-square test

The procedure for testing hypotheses using the chi-square distribution follows the same steps as for the Z and student's t tests.

Step 1: State H_0 and H_a Hypothesis.

When we examine a relationship from sample data, we would like to know if the two variables are independent or related in the population.

Step 2: Select the level of significance e.g. 0.05, 0.02 etc.

Step 3: Select the test statistic. In this case select Chi square donated by x^2.

Given by the formula $x^2 = \sum \left[\frac{(f0-f\theta)^2}{f\theta}\right] \sum \left[\frac{(f0-f\theta)^2}{f\theta}\right]$ with df=k-1

Where k is the number of categories

f0f0 is observed frequency in a particular category

fθfθ is expected frequency inn particular category is observed frequency in a particular category. f_m is expected frequency in a particular category.

Step 4: Formulate the decision rule.

Remember the decision rule in hypothesis testing requires finding the value that separates the region in which we do not reject H_0 from the region of rejection. This value is called the critical value. The critical value for 5 degrees of freedom and 0.05 level of significance is obtained from chi-square tables as 11.070.

Step 5: Compute the value of chi-square and make a decision. In this step, you will either reject or not H_0.

When you reject H_0 you conclude in favor of H_a of a statistically significant relationship

When you accept H_0, you conclude in favor of H_0 that no statistically significant relationship exists.

There are two decision rules you may follow. These are described in sub-section g) below.

Decision rule 1

Find a p-value and compare it to the level of significance alpha (a). P-value are probabilities associated with test statistic (observed x^2) found in step 2 and degrees of freedom (r-1) (c-1) and probability values fund in x^2 square tables.

Find p-value using x^2 tables

Test statistic observed x^2	p-values
Lack for x^2 in the table	Given df = (r-1) (c-1) find probability in the table this might be a range

Decision: If the range of p-value < (a) reject H_0. If range of p-value <(a) accept H_0.

Decision rule 2

Find a critical value of x_σ^2anti compare it to the test statistic critical value X^2 from the table associated with level of significance and degree of freedom.

Decision rule 2:

If test statistic $X^2 > x_\sigma^2$reject H_0 otherwise accept H_0.

Example 19.20

The table below shows observed and expected frequencies in tossing a die 120 times.

Face	1	2	3	4	5	6
Observed frequency	25	17	15	23	24	16
Expected frequency	20	20	20	20	20	20

Required

Test hypothesis that the die is far at 5% level of significant

Solution

Ho: the die is fair

Ha: the die is unfair

Using the formula $x^2 = \sum\left[\frac{(f0-f\theta)^2}{f\theta}\right]\sum\left[\frac{(f0-f\theta)^2}{f\theta}\right]$ where r=2 and c=6.

$$x^2=(\frac{Q1-E1}{E1})^2+\frac{Q2-E2}{E2})^2\frac{Q1-E1}{E1})^2+\frac{Q2-E2}{E2})^2+\ldots\ldots.+\frac{Q6-E6}{E6})^2\frac{Q6-E6}{E6})^2$$

$$= (\frac{23-20}{20})^2 = (\frac{23-20}{20})^2 \quad +(\frac{17-20}{20})^2(\frac{17-20}{20})^2+(\frac{23-20}{20})^2(\frac{23-20}{20})^2+= (\frac{24-20}{20})^2 = (\frac{24-20}{20})^2+= (\frac{16-20}{20})^2$$

$$= (\frac{16-20}{20})^2$$

$$\frac{5^2 5^2}{2020}+\frac{-3^2-3^2}{20\ 20}+\frac{-5^2-5^2}{20\ 20}+\frac{3^2 3^2}{2020}+\frac{4^2 4^2}{2020}+\frac{-4^2-4^2}{20\ 20}=100/20$$

degee of freedom df =(R-1)(C-1)

=(2-1)(6-1)=5

The critical value x_{cc}^{22} for df =5 at αα=5% (0.05) level of significant x^2=11.1 value from x^2 tables since 5.0 < 11.1 *it implies* $x2 < x_c^2 < 11.1$ *it implies* $x2 < x_c^2$

We accept H_0 and conclude that under H_0 the die is fair.

Example 19.21

Given the data in the table below about patients who visited clinics A and B.

	PATIENTS		
CLINIC	SEEN	NOT SEEN	TOTAL
A	282	18	300
B	468	32	500
TOTAL	750	50	800

Required:

Use the given data to test at 1 % confidence level the hypothesis that the patients who were not seen is independent of the clinic they visited.

Solution:

H_0: Patients not seen are independent of clinic visited.

H_a: Patients not seen are dependent of clinic visited

Observed frequency(O)	Expected Frequency (E)	O - E	$(O - E)^2/E$
282	(750)(300)/800 = 281.25	0.75	0.0020
468	(750)(500)/800 = 468.75	- 0.75	0.0012
18	(450)(300)/800 = 18.75	-0.75	0.00300
32	(50)(500)/800 =31.25	0.75	0.0180
TOTAL			0.0512

$\mathbf{x^2 = 0.0512}$

From the tables, **v** = (2-1)(2-1)= 1 degree of freedom

Then, at 1% level of confidence x^2= 6.635

Since 0.0512 < 6.635 we accept the null hypothesis and, therefore, the number of patients not seen does not depend on the clinic visited.

Example 19.22

To compare the effectiveness of two types of baby food A and B, 150 babies were selected at random for the study. Food A was given to 80 randomly selected babies and food B was given to 70 randomly selected babies. At a later time, the health of each baby was observed and classified into 3 categories excellent, average and poor as shown to the table below.

	Excellent	Average	Poor	Sample size
Food A	37	24	19	80
Food B	17	33	20	70
Total	54	57	39	150

From the frequency counts recorded test the hypothesis that there is no difference between the qualities of the two types of food at 5% level of significance

Required:

	Excellent	Average	Poor	Sample size
Food A	37	24	19	80
Food B	17	33	20	70
Total	54	57	39	150

Solutions:

H_0: There is no difference in quality between the two types of food.

H_a: There is a difference between the quality of the two types of food.

The value of value of $\frac{(0-E)^2 (0-E)^2}{E \quad E}$

	Excellent	Average	Poor
Food A (0)	37	24	19
Expected	28.8	30.4	20.8
Food B (0)	17	33	20
Expected	25.2	26.6	18.2

. $X^2 = 8.224$

Degrees of freedom for X^2 = (r-1) (c-1) =(2 - 1)(3 - 1)=2 At 5% level a =0.05 df =2 Xc = 5.99. This implies that 8.224>5.99 which means $X^2 > X^2c$. This implies H_0 is rejected. Conclude that there is a significant different between the quality of the two types of food.

Examples 19.**23**

Consider the number of employees who belong to the companies MTN, TIGO and AIRTEL. Test whether there exists any association between the characteristic by party upon which classifications is made at a=0.05 level of significance.

	Manual employment	Non manual
MTN	18	19
TIGO	46	14
AIRTEL	11	13
TOTAL	75	46

Note: This classification is in 2 ways according to a political party and to a type of employment This is an example of a contingency table 3x2 type. This table gives a means of testing if there exists any association between its characteristics upon which classification is made.

Solution:

H_0: There is no relationship between characteristic upon which people are employed.

H_a: There is a relationship between the characteristic upon when people are employed.

Degree of freedom = (3-1) (2-1) =2 expected frequencies

expected frequency

Expected MTN's engaged in manual $=\frac{37 \quad 37}{121121}$x75=22.9

Similarly others are obtained placed in the table as shown below

	Expected is manual	Expected is non-manual	Total
MTN	22.9	14.1	37
TIGO	37.2	22.9	60
AIRTEL	14.9	9.1	24

The expected frequencies are calculated and indicated in the table below.

Total	75	46	121

From this table we calculate

$X^2 = \sum \frac{(0-Eo)^2}{Eo} = 11.1 \sum \frac{(0-Eo)^2}{Eo} = 11.1$

X^2c at $\alpha = 0.05$ df $= 2, X2 = 5.99 \alpha = 0.05$ df $= 2, X2 = 5.99$

Hence 11.1> 5.> 5.9 which implies x^2>> X^2c

We reject Ho and accept Ha , that there is a relationship between employment and the party one belongs to

Examples 19.24

A random sample of 735 employees in a certain institution is classified by two characteristic namely housing (do they own or rent a house) and car they drive (Toyota ,Nissan or Isuz) as shown in the table below

	Car			
Housing	**Toyota**	**Nissan**	**Isuzu**	**Total**
Own	200	160	40	400
Rent	90	168 0	77	335
Total	290	328	117	735

Required:

Test at 5% level of significance the assertion that the classifications are independent.

Solution:

H_o - the classifications are independent. H_a - the classifications are dependent.

o	E	O-E	$(O - E)^2/E$
200	157.82	42.18	11.2733
90	132.18	-42.18	13.4601
160	178.50	- 18.50	1.9174
168	149.50	18.50	2.2893
40	63.67	-23.67	8.7996
77	53.33	23.67	10.5057
TOTAL			48.2454

x^2=48.2454

From the tables; v = (2-1)(3-1) = 2 degrees of freedom

Then at 5% level of significance, the approximate x^2= 5.991, since 48.2454 >5.991, we reject the null hypothesis, hence, conclude that owning or not owning a house depends on the car one drives.

Example 19.25

A bag contains white, blue and yellow identical marbles. Forty-eight marbles are randomly picked from the bag, each marble immediately replaced after picking and noting its colour. The table below shows the results obtains.

Colour	White	Blue	Yellow
No. of marbles	11	23	14

Required:

Determine at 5% level of significance whether the bag contains equals number of marbles of different colours.

Solution:

H_0: different colors of marbles are of the same number.

H_a : the marbles of different colours are of different number.

Expected frequency $\frac{11+23+14}{3}\frac{11+23+14}{3}$=16

0	E	O-E	$(0-E)^2/E$
11	16	-5	1.5625
23	16	7	3.0625
14	16	-2	0.2500
Total 48	48	0	0.0000
SUM			4.8750

x^2= 4.875

From the x^2distribution, the v = 3—1 =2. At 5% level of significance x^2=5.991.

Since 4.875 <5.991, the null hypothesis is accepted, hence, we conclude that the various colours of marbles are of equal number.

Example 19.**26**

Sixty students on a trip were asked which yoghurt flavours they like out of the three flavours: vanilla, straw berry and chocolate. The responses were recorded as

Flavour	Number
Vanilla	17
Straw berry	24
Chocolate	19

Required

Determine whether students prefer any particular flavour at a 0.05 level of significance.

Solution

Flavour	Observed	Expected
Vanilla	17	20
Straw berry	29	20
Chocolate	19	20

Self-test questions

Z- test Question 1

State the null and alternative hypotheses for:

a) the mean life span of a Rwanda n is 48 years;
b) the mean number of girls in the quantitative techniques class is not more than 30.
c) the mean weight of babies born in the health centre is 3.8kg.

Solution:

a) H_0: n = 48 years

H_a: fi <48 years

b) H_0; n <30 girls

H_a: n >30 girls

c) H_0: **u** = 3.8 kg

H_a: n 3.8 kg

Question 2

A judge decides to acquit all defendants, regardless of the evidence. State the type I and type II errors associated with this policy?

Solution:

Type 1 error releasing a guilty defendant.

Type 2 error releasing defendants without hearing.

Question 3

The waiting time for a service after placing an order at hotels in a certain town is on average 19 minutes. A random sample of 18 hotels showed a mean waiting time of 17.78 minutes with a standard deviation of 0.41 minutes.

Required:

a) State the hypotheses and the decision rule.

b) Use a t-statistic, test at 5% if the true mean is less than the specification.

Solution

Z calculated = 0.1179 $< t\ from\ tables = 1.740 < t\ from\ tables = 1.740$

Accept Ho, Average waiting time is 19 minutes.

Question 4

Faced with rising printing costs, an organisation issued a guideline that printing by a Risograph should be for 10 pages or more. The firm examined 35 randomly printed documents that yielded a sample mean of 14.44 pages with a standard deviation of 4.45 pages.

Required:

At the 0.1 level of significance, find whether the true mean is greater than 10 pages. Solution:

Z0.05 = 1.65 from tables, Z calculated = -0.1657

Since -0.1657 < 1.65 accept H_0

Printing by isograph should be restricted to 10 or more pages.

Question 5

In an audit theory exam, the mean score for 43 male students is 21.2 and standard deviation is 4.9. The mean score for 56 female students is 20.9 and standard deviation is 4.6.

Required:

Test at a=0.01, to see whether the male and female students have equal audit theory exam score **t -test.**

Solution:

The critical value of Z ȳȳ$_{(}\alpha\alpha_{=0.01})$ = 2.58., Z = 0.31 Since 0.31 < 2.58 accept Ho. The male and female have equal Audit theory marks.

Question 6

A carpet manufacturer is studying differences between two of its major outlet shops in Kigali. The company is particularly interested in the times it takes before customers receive the carpets ordered from the factory. The data concerning the delivery times for most popular type of carpets is summarized in the table below.

	ShopA	Shop B
$\bar{x}$	34.3 days	43.7 days

S	2.4 days	3.1days
N	41	31

Required:

At 0.01 level of significance, is there evidence of a difference in the average delivery times for the two shops.

Question 7

The data below represents tuition charges in the training colleges in western and Eastern Rwanda.

Western Rwanda(10,000 Frw)			**Eastern Rwanda(10,000 Frw)**		
10.5	8.9	9.6	7.9	10.6	8.4
10.1	9.3	9.1	8.6	10.1	9.2
10.0	9.7	11.2	9.1	8.5	10.7
11.0	10.4	10.5	9.3	7.5	9.5
9.8	10.0	9.9	8.8	9.3	9.8

a) For each region calculate the mean and standard deviation.
b) Suppose you have a cousin who wishes to join, advise him/her which better institution is. Give two reasons for your choice.
c) Is there evidence of major difference in tuition fee in the institutions at 1% level of significance in the two regions.

Question 8

The following is data for the cost in "1000 s" Rwandan Francs of shampoo of a random sample of 31 shampoo containers labelled normal hair and 29 shampoo containers labelled fair hair.

Normal hair					**Fair hair**				
79	63	19	9	37	69	9	23	22	8
84	20	16	55	69	12	32	12	18	74
23	14	9	87	44	19	63	49	37	55
13	16	23	20	64	85	44	87	17	11
28	18	32	81	85	23	50	65	51	35
47	50	8	13	21	14	20	28	**8**	

Required:

a) Calculate the mean and standard deviation for the two types of shampoo.

b) Is the evidence of a difference in the average cost between shampoos label Normal hair and those labelled for fair hair at α= 0.05.

Use (i) t-test and (ii) z-test to arrive for your conclusion.

c) If you were asked to write an article for a magazine comparing the two types of shampoo

What conclusions would you make?

Question 9

A random sample of processing times by two computers Dell and Sharp were recorded in the table below.

Dell (seconds)	**Sharp (seconds)**
0.08	3.3

4.4	7.5
1.3	1.3
3.9	6.3

Required:

a) i) Determine the mean and variance of the processing power.

ii) State the null and alternative hypothesis.

iii) Test whether the population means differ significantly using t - test at 10% level.

b) Assume population standard deviation are known to be σ = 1.1 for Dell and σ_2 - 1.2 for Sharp.

Test whether the sample means are significantly different using σ = 0.05.using z - test.

Chi-square test

Question 10

Outline and explain with an example the steps involved in-a X^2 test of association, the type data on which such a test is used and the way in which the valine of X^2 is interpreted.

Question 11

In a survey to assess the contribution of the professional qualification on the practice of a certain profession, a random sample of 200 people was used of which 200 were profession while 200 had the basic qualification. After administering a promotional interview 40 professional and 10 non-professionals were promoted.

Required:

a) Present the above data in a contingency table.

b) Test at 1 % confidence level the hypothesis that the interview results were independent of the qualification.

Solution:

a) Contingency table

	Promoted	Not promoted	Totals
Skilled	140	60	200
Unskilled	110	90	200
Total	250	150	400

b) Calculated value $\mathbf{X^2}$ **=** 34 > table x^2 = 6.65

Reject null hypothesis.

Question 12

In investigating the association between* background and performance of students in a certain university, the following data was collected from a random sample of 90 students.

1	**Performance**	
Social class	**Above average**	**Below average**
Upper	12	5
Middle	30	12
Lower	25	6
Total	67	23

Required:

Test whether there is association between the background and the performance at 5% level of significance.

Solution:

Calculated value X^2 = 54.42 > table X^2 = 5.991. Reject null hypothesis. There is no association.

Question 13

In a random sample of 120 miners in a certain cobalt mine, the statistics show that 40 were from the eastern region, 55 from the western region and 25 from the central region.

Required:

Test at 5% level of significance the hypothesis that the miners are distributed in the ratio 3A 3 to the respective regions.

Solution:

Calculated value X^2 = 4.8264 < table X^2 = 5.991

Accept null hypothesis. The miners in the three groups are in the ratio 3:4:3

Question 14

At Smart University, a random sample o£20Cffinalists BBA and BCOM and BSC were selected to ascertain whether they intended to apply for CPA professional course. Their responses were recorded in the table below.

Plan	Degree course			
	BBA	**BCOM**	**BSC**	**Total**
Apply	5	8	7	20
Not apply	67	49	24	140
Undecided	8	13	19	40
Total	**80**	**70**	**50**	**200**

Required:

Test the hypothesis that degree program plans are independent of post graduate professional course at 0.05 level of significance.

Solution:

At 0.05 level of significance, H_0 is rejected, because the differences between observed and expected values are too large to be by chance. We conclude that the undergraduate course; attended affects the postgraduate course plans.

Question 15

Applicants for welfare aid are allowed on appeal process when there is a feeling of being unfairly treated. At one learning centre the applicant may choose self-representations or representation by attorney. The appeal may result in an increase or decrease or no aid given. Court records of 320 appeal cases provide the following data.

Type of recommendation	Amount and aid		
	Increased	**Uncharged**	**Decreased**
Self	59	109	17
Attorney	70	63	3

Required:

Test whether the patterns of appeals decision are significantly different between the two types of representation at a=0.05

Solution:

Observed value $X^2 = 15.73$, df = 2,a = 0.05 $X^2c = 5.991$. Hence, patterns of appeals decision are significantly different.

Accept null hypothesis. The miners in the three groups are in the ratio 3:4:3.

CHAP 20.

INDEX NUMBERS

20.1. Study objectives

By the end of this chapter, you should be able to: describe the term index numbers; explain factors to consider when constructing index numbers; explain the importance and limitations of index numbers; distinguish between simple and weighted index numbers; compute and interpret index numbers; distinguish between fixed base and chain base methods; compare Laspeyre's and Paasche's price and quantity index numbers; and calculate cost of living, retail, stock, etc. index numbers.

Definition 20.1: An index number is a value indicating average change in magnitude of price, quantity, or value of a group of items over a period of time. The specified period or point on which the calculation is based is referred to as the base year or period, whereas the period or point for which the index number is computed is called the current year or period. Index numbers are usually expressed as percentages.

Index numbers provide a standardized way of comparing the values of commodities (such as prices, volume of output, wages, etc) over time. They are used extensively, in various forms in business, commerce and government departments.

Construction of index numbers

The following factors are taken into consideration:

a) Purpose of the index numbers

The purpose of the index numbers guides on choice of the technique or method of construction and this ensures that the results obtained are relevant.

b) Selection of commodities

While selecting the commodities, a great care and skill should be used as the proper selection of the commodities helps achieve the purpose of the construction of index numbers.

c) Price quotations

It may not be possible to collect the prices for selected commodities from all the places where they are marketed. A sample of markets will, therefore, have to be selected. The criteria of selection will be to choose places where given commodities are marketed in large volumes. It is just possible that a sample may serve the purpose for many commodities rather that one commodity.

d) **Choice of the base period**

The base, year should be one with neither very low prices nor very high prices, usually referred as a normal year. However, it is probable that no one year is sufficiently normal to be a good basis for comparison.

e) Choice of the average

When studying the index number of a single commodity, average is not needed. But in a case involving more than one commodity the price relatives are computed and averaged. In this case the averages are to be used and it is to be decided which average is used as there are different measures of averages. Generally, the following averages are used for this purpose:

- arithmetic mean;
- median; and
- geometric mean.

f) Choice of proper weights

Index numbers include many commodities which are not equally important.

Therefore, it is important to give weights according to the importance of different commodities keeping in mind the purpose of the index numbers. Weighting shall be done with careful consideration and skill after a detailed study of the purpose of index numbers.

20.2. Simple index numbers

Types of simple index numbers

When the construction of index numbers involves a single variable, these are called simple index numbers. There are three general types of index numbers classified in terms of their variables.

- Price index numbers. These are index numbers which measure changes in prices of commodities over a period of time.
- Quantity index numbers. These are index numbers that measure changes in production and output.
- Value index numbers. These are index numbers which measure changes in :ie value of the various commodities and activities.

There are two methods for the construction of simple index numbers, that is. fixed case method and chain base method.

Price index numbers

The index numbers of prices are important because they show that the value of money is fluctuating, that is appreciating or depreciating accordingly as the index numbers of prices are rising of failing. A rise the index number of prices will signify the deterioration in the value of money.

Time series relatives

Construction of price index numbers

a) **Fixed Based Method**

A fixed based method for constructing simplex index numbers uses the same base period hence the name fixed base. The method expresses the price of subsequent years as price relatives of the base year, given as $\frac{PcPc}{PoPo}$, where Pc is the price in the current year (period), Po is the price index in the base year. When comparing prices of one item between two different periods of time, the price index is calculated by the fixed base formula:

Price index =$\frac{Pc}{Po}$ **x100**$\frac{Pc}{Po}$ **x100**

Price relative is also considered as the price index of a commodity.

Example 20.1

The price of carton of water was Frw 10,000 in 2009 while in 2011 the price was Frw 10,200.

Required:
i) Using 2009 price as base period, calculate the price index.
ii) Interpret the price index.
Solution:

Price index $=\frac{P_c}{P_o}\ \mathbf{x100}\ \frac{P_c}{P_o}\ \mathbf{x100} = \frac{10200}{10000}\ \mathbf{x100}\ \frac{10200}{10000}\ \mathbf{x100}$=102
ii) The price of a carton of water rose by 2% from 2009 to 2011.
It should be noted that the base year (or period) is always specified and the base year index=100.
Example 20.2

From the data below, giving the price of sugar over a period of six years:

Year	2001	2002	2003	2004	2005	2006
Price of Sugar /kg	800	1000	1250	1800	2000	2500

Required:
Compute the price indices, taking 2001 as base year.
Solution:

Year	Price of sugar/kg	Price index$(\frac{P_c}{P_o}\ \mathbf{x100}$ $\frac{P_c}{P_o}\ \mathbf{x100})$
2001	800	$(\frac{800}{800}\ \mathbf{x100}$ $(\frac{800}{800}\ \mathbf{x100}$)=100.0
2002	1000	$\left(\frac{1000}{800}\ \mathbf{x100}\right) = 125.0$
2003	1250	$(\frac{1250}{800}\ \mathbf{x100})$ $(\frac{1250}{800}\ \mathbf{x100})$ =156.0
2004	1800	$(\frac{1800}{800}\ \mathbf{x100})$ $(\frac{1800}{800}\ \mathbf{x100})$ =225.0
2005	2200	$(\frac{2200}{800}\ \mathbf{x100}) = 275.0$
2006	2500	$\left(\frac{2500}{800}\ \mathbf{x100}\right) = 312.5$

b) **Chain base method**

The chain base is a method for constructing simple index number that uses different base periods from year to year . This method shows whether the rate of change is rising, filling, or constant as well as the extent of change from year to year.
The previous period (year) is usually taken as the base period for the subsequent period (year) when calculating the index.

In this method, Price index number $=\frac{\textit{Price of the current year}}{\textit{Price of the previous year}}\ \mathbf{x100}\ \frac{\textit{Price of the current year}}{\textit{Price of the previous year}}\ \mathbf{x100}$

Example 20.3
The table shows the price variation of a certain commodity from 2001 to 2006.

Year	2001	2002	2003	2004	2005	2006
Prices (Frw)	1200	1250	1400	1500	1350	1600

Required:

Construct the chain base index numbers from the following data.

Solution:

In 2001, there is no chain base index number because the price for the previous year, 2000 is not given.

To find the chain base index number for 2002,use the formula:

Price index number $=\frac{Price\ of\ the\ current\ year}{Price\ of\ the\ previous\ year}x100\frac{Price\ of\ the\ current\ year}{Price\ of\ the\ previous\ year}x100 = \frac{1250}{1200}x100\frac{1250}{1200}x100$ =104.17

Use the same formula to obtain the subsequent chain base index numbers as summarized in the table below:

Year	Price (Frw)	Chain base index number
2001	1200	-
2002	1250	$\left(\frac{1250}{1200}x100\right) = 104.17$
2003	1400	$(\frac{1400}{1250}x100) = 112.0$
2004	1500	$(\frac{1500}{1400}x100) = 107.14$
2005	1350	$(\frac{1350}{1500}x100) = 90.00$
2006	1600	$(\frac{1600}{1350}x100) = 118.52$

Merits of chain index method:

Provides a direct comparison between each year and the preceding year and it is in such terms that a businessman often thinks

Allows the addition of new commodities, removal of old commodities or the substitution of one commodity for another.

It is possible to change the geographical coverage or the weight of a commodity to meet changing conditions

An index with a fixed period can be computer by the product of link relatives.

Demerits of chain index method:

The computational procedure of chain base index numbers is relatively cumbersome.

If an error is committed during the chaining process, it will be carried through the entire series.

In the absence of data between two years, chain base index numbers cannot be computed.

Weighted index numbers

This method is used when commodities selected are not of equal importance to the consumers. The weighting system is adopted by assigning appropriate weights to the different commodities. Sometimes, the quantities used are taken as weights. The use of weights makes the index numbers more reliable. Weighted index numbers are those constructed taking weights into consideration.

Methods used for constructing weighted index numbers:

Weighted aggregate method.

Weighted average of price relative method.

20.3. Weighted aggregate method

An index is called a weighted aggregate index when it is constructed for an aggregate of items(prices) that have been weighted in some way (by corresponding quantities produced, consumed or sold) so as to reflect their importance. Some of the methods commonly used include the following:

Laspeyre's method
Paasche's method
Fisher's ideal method
Marshall Edgeworth method.

Laspeyre's method

In this method, the base year's quantities are taken as weights. The formula used is given by:

$I_p = \frac{\sum Pc\,Qo}{\sum Po\,Qo}\,x100 \frac{\sum Pc\,Qo}{\sum Po\,Qo}\,x100$

Paasches method

In this method, the current year quantities are taken as weights. The formula used is given by:

$I_p = \frac{\sum Pc\,Qc}{\sum Po\,Qc}\,x100 \frac{\sum Pc\,Qc}{\sum Po\,Qc}\,x100$

Example 20.4

Give the date in the table below:

	1979		1986	
	Price (USCHs)	Quantity (Bags)	Price (Frw)	Quantity (Bags)
Maize	65	20	135	30
Wheat	95	8	160	7
Beans	150	5	320	8

Required:

Taking 1979 as the base year calculate the following index numbers forb1986:

i) Laspeyre's.

iii) Paasche's

Solution:

	1979		**1986**				
	P_0	q_0	P_1	q_1	P_1q_0	P_1q_1	P_0q_1
Maize	65	20	135	30	2700	4050	1950
Wheat	95	8	160	7	1280	1120	665
Beans	150	5	320	8	1600	2560	1200
					$\sum P_1q_0=5580$	$\sum P_1q_1=7730$	$\sum P_0q_1=3815$

i) Laspeyre's index numbers, $I_p = \frac{5580}{2810}x100 = 198.6 \frac{5580}{2810}x100 = 198.6$

ii) Paasche's index number, $I_p = \frac{\sum P1q1 \sum P1q1}{\sum P0q1 \sum P0q1}$x100=$\frac{77307730}{38153815}$ x100 =202.6

Comparison of Laspeyer's and Paasche's methods

From a practical point of view, Laspeyre's index is often preferred to Paasche's because in Laspeyres index weights (q_o), the base year quantities do not change from one year to the next.

On the other hand, the use of Paasche's index requires the continuous use of new quality weights for each period considered and in most cases these weights are difficult and expensive to obtain.

Weighted average of price relatives method

The weighted average price index number is given by the formula:

$\frac{\sum(\frac{Pc}{Po})w \sum(\frac{Pc}{Po})w}{\sum W \quad \sum W}$ x 100, where $\frac{p_1 p_1}{p_0 p_0}$ = the price relative (price index)
w = weight.

Note: This index may be referred to as just the weighted price index.

Example 20.5

Given the market basket of goods and services and their prices below:

Item	2009 Price (Frw)	2011 Price (Frw)	Weight(w)
Entertainment	55,000	60,000	1
Transport	1,6000	20,000	2
Rent	48,000	52,000	3
Food	60,000	58,000	4

Required:

Calculate the weighted average price index for the given market basket of goods and services.

tem	Po	Pc	W	$w\left(\frac{p_1}{p_0}\right)$
Entertainment	55,000	60,000	1	$1\left(\frac{60000}{55000}\right) = 1.0909$
Transport	16,000	20,000	2	$2\left(\frac{20000}{16000}\right) = 2.5000$
Rent	48,000	52,000	3	$3\left(\frac{52000}{48000}\right) = 3.25000$
Food	60,000	58,000	4	$4\left(\frac{58000}{60000}\right) = 3.8667$
			£w=10	$\sum w\left(\frac{p_1}{p_0}\right) = 10.7076$

The weighted average index = $\frac{(\frac{Pc}{Po})}{\sum w}$x100$\frac{(\frac{Pc}{Po})}{\sum w}$x100 = $\frac{10.7076}{10}$x100$\frac{10.7076}{10}$x100=107.076

Note: In case the weights are not given, they are calculated using the expression

$$W = p_0 q_o$$

Example 20.6

The table below shows the units of items used by a certain restaurant and their unit cost prices.

Item	Price 1961 (Frw)	Price 1966 (Frw)	Quantities units

Chicken	25	50	100
Beef	20	40	30
Mushroom	20	30	50
Fruit	10	18	100
Greens	30	45	50

Required:

Compute the index number of 1996 from thee data above by using weighted average of price relative method

Solution:

Item	q_0	Po	P_1	$\frac{p_1}{p_o}$	$w(P_0q_0)$	$\frac{PcPc}{--}w$
Chicken	100	25	50	200	2,500	500,000
Beef	30	20	40	200	600	120,000
Mushroom	50	20	30	150	1,000	150,000
Fruit	100	10	18	180	1,000	180,000
Greens	50	30	45	150	1,500	225,000
					$\sum w = 6600$	$\sum \frac{P_c}{P_0} w = 1{,}175{,}000$

The weighted index number for 1966=$\frac{\sum iv \sum iv}{\sum v \sum v}$=$\frac{1175000 1175000}{6600 \quad 6600}$ = 178.03

Advantages of weighted average of price relative method

Index numbers constructed by the average of price relatives method, all of which have the same base, can be combined to form a new index.

It is the only appropriate method when an index number is computed by selecting one item each from the many sub-groups of items and the values of each sub-group used as weights.

When a new commodity is introduced to replace the one formerly used the relative for the new item may be spliced to the relative for the old one. using the former value weights.

The price or quantity relatives for each single item in the aggregate are. in effect, themselves a simple index that often yields valuable information for analysis

Quantity index numbers

Quantity index numbers measure and permit comparison of the physical volume of goods produced or distributed or consumed. The construction of quantity index numbers is similar to that of price index numbers. In this case however, the interest is to measure the changes in quantities.

Fixed base method

When comparing quantities of one item consumed between two different periods of time, the quantity index is calculated by the fixed base formula quantity index =$\frac{q_c q_c}{q_o q_o}$ x 100

The quantity relative is also considered as the quantity index of a commodity.

Example 20.7

A certain soft drink whole seller sold 22,000 crates in 2009. If the sales in 2010 were 18,000 crates.

Required:

i) Using 2009 as base period, calculate the quantity index for 2010.

ii) Interpret the quantity index.

Solution:

i) Quantity index $\frac{q_c}{q_o} \text{ x } 100 = \frac{18000}{22000} \text{ x } 100 = 81.82 \frac{q_c}{q_o} \text{ x } 100 = \frac{18000}{22000} \text{ x } 100 = 81.82$

ii) The sales of soft drinks decreased by 18.18% from 2009 to 2010.

Example 20.8

The table below shows the average output of a small factory, milling a-d packing cereal flour for local consumption and export to the regional market over a period of four years, from 2008 through 2011.

Year	2008	2009	2010	2011
Output (tonnes)	150	175	160	182

Required:

i) Calculate the quantity index for each of the years taking year 2008 as the base

ii) Interpret each index number computed.

Solution:

I) Year	Quantity(q_o)	Quantity (q_c)	Index number ($\frac{q_c}{q_o}$) x 100 $\frac{q_c}{q_o}$) x 100
2008	150	150	$\frac{150}{150} \text{ x } 100$ =100.00 $\frac{150}{150} \text{ x } 100$
2009	150	175	$\frac{175}{150} \text{ x } 100$ =116.67 $\frac{175}{150} \text{ x } 100$
2010	150	160	$\frac{160}{150} \text{ x } 100 =$ 106.67 $\frac{160}{150} \text{ x } 100 =$
2011	150	142	$\frac{142}{150} \text{ x } 100 =$ 94.67 $\frac{142}{150} \text{ x } 100 =$

ii)The output increased in 2009 and 2010 by 16.67% and 6.67% respectively.

However, that of 2011 decreased by 5.33%.

Chain base method

Using this method, quantity index number=$\frac{\text{quantity of the current year}}{\text{quantity of the previous year}} \text{ x } 100 \frac{\text{quantity of the current year}}{\text{quantity of the previous year}} \text{ x } 100$

Example 20.9

Give the data in the table in Example 7.

Required:

Determine quantity index for each of the year using the chain base method.

Solution:

Year	Quantity(q_o)	Quantity(q_c)	Index number $(\frac{q_c}{q_o})$ x 100 $\frac{q_c}{q_o})$ x 100
2008		150	
2009	150	175	$\frac{175}{150}$ x 100 $\frac{175}{150}$ x 100=116.67
2010	175	160	$\frac{160}{175}$ x 100 $\frac{160}{175}$ x 100=91.43
2011	160	142	$\frac{142}{160}$ x 100 $\frac{142}{160}$ x 100=88.75

Weighted quantity index number

There are two formulae for finding the weighted quantity index numbers. These formulae represent the quantity index in which the quanties of the different commodities are weighted by their prices. However, any other suitable weights can be used.

Laspeyre's method

$$I_q = \frac{\sum QcPo}{\sum QoPo} x100 \frac{\sum QcPo}{\sum QoPo} x100$$

Where q_o is the base year quantity, q_c is the current year quantity, p_o is the base year price,

I_q is the quantity index number, Paasche's method

$$I_q = \frac{\sum q_c p_c}{\sum q_o p_c} x100$$

Where pc is the current year prices

q_O is the base year quantity qc is current year quantity

q_c is the base year price

I_q = the quantity index number

Example 20.10

Given the date in the table

	2003		2006	
Commodity	**Price**	**Value**	**Price**	**Value**
A	8	80	10	110
B	10	90	12	108
C	16	256	20	340

Required:

Complete the 2006quantity index by the Paascher's method taking 2003 as the base year.

Solution:

	2003		**2006**			
Commodity	**p_0**	**q_0**	**p_1**	**q_1**	**$q_0 q_1$**	**$q_1 p_1$**
A	8	10	10	11	100	110
B	6	9	12	9	108	108
C	16	16	20	17	320	340
					$\sum q_0 q_1$=528	**$\sum q_1 p_1$=558**

Using the Paasche formula, the quantity index:

$$I_q = \frac{\sum q_c p_c}{\sum q_o p_c} x100$$

$I_q = \frac{558}{528} x10I_q = \frac{558}{528} x100$ =105.6818

20.4. Consumer price index numbers

These are also known as cost of living index numbers. A cost of living index number measures the changes in the purchasing power of the consumers. The cost of living index numbers are especially designed to study the effect of changes in prices.

A rise in the cost of living index number shows the fall in the standard of living.

In order to measure the effect of rise and fall of the prices of different commodities on th general standard of living, different index numbers are constructed for different groups of people.

Uses of the consumer price index numbers

Used in wage negotiations and wage contracts. Automatic adjustments of wage and allowance component of wages are governed in many countries by such indices.

At government level, these index numbers are used for wage policy, price policy, rent control, taxation and general economic policies.

They are also used to measure changing purchasing power of the currency, real income, etc.

Index numbers are also used for analyzing markets for particular kinds of goods and services.

Construction of a consumer price index

The following should be considered in construction of a consumer price index numbers:

Decision about the class of people for whom the index is targeting.

It is absolutely essential to decide clearly the class of people for whom the index is meant, that is, whether it relates to industrial workers, teachers, officers, etc. the scope of the index must be clearly defined.

Selection of commodities.

This involves preparing a list of commodities commonly consumed (in terms of tastes, preference, customs, etc.)

Obtaining price quotations:

Price quotations should be obtained from the localities in which the class of people concerned reside or from where they usually make their purchases.

Weightings:

In order to calculate index numbers, the prices or price relatives must be weighted. The need for weighting arises because of the relative importance of various items for different classes of people.

Choice of the base year and the method to apply: Sometimes it is a challenge to select a year with stable prices.

Method of constructing the index

The cost of living index numbers may be constructed by applying any of the following methods: Aggregate expenditure method, that is, the Laspeyre's method. In this case:

Consumer price index I_p = $\frac{\sum q0p0 \sum q0p0}{\sum qoPO \sum qoPO}$

Note: This method is not popular method for constructing consumer price index.

Family budget method, weighted average price relatives method. In this case: Consumer price index= $\frac{\sum(\frac{pc}{po})w \sum(\frac{pc}{po})w}{\sum w \quad \sum w}$

This is most commonly used method for constructing consumer price indices or cost of living indices.

Example 20.11

Prices per unit of the items forming consumption bundle of an average middle family in two periods and percentage of total family budge allocated to the given in the following table:

		Food	Rent	Clothing	Utilities	Entertainment
	Expenditure (%)	**35**	**15**	**20**	**10**	**20**
	Period 0	**150**	**50**	**100**	**20**	**60**
Price (in)	Period 1	**174**	**60**	**125**	**25**	**90**

Required:

Compute an appropriate index number and comment on the result

Solution

The appropriate index number Kere would be the consumer Price index number

Items	Po	Pc	-X 100 **(I)**	v	**IV**
Food	150	174	116	35	4.060
Rent	50	60	120	15	1,800
Clothing	100	125	125	20	2,500
Utilities	20	25	125	10	1.250
Entertainment	60	90	150	20	3,000
				$\sum V = 100$	$\sum IV = 100$

Therefore the consumer price index =$\frac{\sum IV \sum IV}{\sum V \ \sum V}$=12,610/100 =126.1

Comment: An increase of **26.1** percent occurred in the consumer price index in the current year.

Disadvantages of consumer price index numbers

It is practically difficult to clearly demarcate one class of people from another. Consumer price index numbers construction involves sampling of goods an services. Sampling errors and biases may affect the index number. It is difficult to collect prices of certain goods. For example, prices of fashion goods.

It is also difficult to eliminate the effect of changes in quality and grade of goods; and services purchased. The consumer price index numbers cannot be used for comparing the price changes in consumer goods and services in two localities or in two households in the same locality as no two households can be homogeneous, that is. They can neither have precisely the same pattern of consumption nor precisely the same basket of goods and services.
Note: A consumer price index numbers may not be wholly relied upon because it is an imperfect measure as evident above.

Stock Exchange Index

The stock exchange index number indicates the level of investment in stock exchange securities as compared to the base period. These index numbers are a measure relative change from one period another. Stock index numbers are constructed **to** measure the general price movement in listed shares of a stock exchange. They are, therefore, important instruments in measuring the market dynamics.

Note: Consumer. Stock exchange and Retail indices are sometimes refer to **as** Published indices.

Uses of index numbers

- The price index numbers are used to measure changes in a particular group of prices and help in comparing the movement in prices of one commodity with another. They are also designed to measure the changes in the purchasing power of money.

- Index numbers of industrial production provide a measure of change in the level of industrial production in a country.

- The quantity indices show the rise or fall in the volume of production, volume of exports and imports, etc.

- The import and export price index numbers are used to measure the changes in the terms of trade of a country, which is the ratio of import to export prices. They are also used to forecast business conditions of a country and to discover seasonal fluctuations and business cycles.

- The consumer price index numbers indicate the movements in retail prices of consumer goods and services. These movements in prices help government in formulating its policies and in taking appropriate economic measures. They can be used to re-adjust the wages and to adopt measures of relief by granting allowance and bonus to their employees to meet the increases costs by the industrial and commercial establishments as well as mills. They are also used to deflate the gross national product and wages to arrive at the real values of the national product and real wages.

- Index numbers are also used to measure enrolment changes, intelligence quotients and the performance of students.

Limitations of index numbers

- It is not practicable to price all the goods and services as well as to take into account all changes in quantity and price of products.

- The construction of index numbers is based on sampling may contain sampling errors.

- The choice of a base period may be difficult as few periods car be regards: as normal for all segments of the economy.

- The results obtained by different methods of construction may not quite agree Comparisons of changes in variables over long periods are not reliable. Index numbers may not be suitable for all purposes. The users are strongly ac. sec to understand the purpose for the index number.

Self-test questions

Question 1

Describe briefly the following term§ as applied to index numbers:

a) Simple index numbers.
b) Weighted index numbers.
c) Quantity index numbers.
d) Fixed base method.
e) Chain base method.

Question 2

Explain the short-comings of consumer index numbers.

Solution:

An explanation of short-comings of consumer index numbers can be found in Chapter 17, section 15d).

Question 3

a) Define the term index numbers.
b) Explain the uses and imitations of index numbers.

Solution:

a) The definition can be found in Chapter 17. section 1.

b) The uses and limitations can be found in Chapter 17, section 18

Group	Index for 1975	Expenditure (%)
Food	650	40
Rent	400	20
Utilities 1	350	5
Entertainment	150	15
Others	300	20

Question 4

a) What is a price relative?

b) Explain the use of weighting in index numbers.

Question 5

Given the table of data below calculate the cost of living index

Item	Price relative	Weighting
Tea	100.8	3
Clothing	107.2	8
Utility	100.6	8
Shelter	115.3	6
Entertainment	111.2	11
Transport	112.5	12
Food	103,4	25

Solution

The cost of living index is 107.1

Question 6

c) The index numbers calculated on the chain base method are given for the three years, to the nearest integer as in the table below:

2009	2010	2011
100	107	108

Required:

Calculate the 2011 index taking 2009 as the base year.

Solution:

The 2011 index is 116, to the nearest integer.

Question 7

The table below shows data for 2014.

Item	Weighting	Percentage increase on 2013 costs
Clothing	10	5
Equipment	10	7
Groceries	15	8
Transport	20	5
Labour	45	14

Required:

Determine the index of service cost for 2014 taking 2013 as the base year.

Solution:

The index of service cost for 2014 is 109.7 (to 1 dp).

CHAP 21.

INTRODUCTION TO FINANCIAL MATHEMATICS

21.1. Percentages and Ratios, Simple and Compound Interest, Discounted Cash Flow

21.1.1. Percentages

A percentage is really a fraction where the denominator is 100, and we use the symbol %. To convert a fraction or a decimal to a percentage, simply multiply by 100.

$\frac{7}{100}$ $becomes$ $\frac{7}{100}x100 = 7\%$

$\frac{3}{10}$ $becomes$ $\frac{3}{10}x100 = 30\%$

0.8 $becomes$ $0.8x100 = 80\%$

0.15 $becomes$ $0.15x100 = 15\%$

$\frac{1}{3}$ $becomes$ $\frac{1}{3}x100 = 33\frac{1}{3}\%$

To find the percentage of a number, we simply convert the percentage to a fraction and multiply:

8% OF RWF10,000=$\frac{8}{100}X10000 = 800$

21.1.2. A ratio

A ratio is another form of fraction just like a percentage, decimal or common fraction. It is a relationship between two numbers or two like values. Consider the following situation concerning a small town:

Employed	3,000
Unemployed	1,000
Total workforce	4,000 (available for employment)

This can be expressed in several ways:

¼ of the workforce is unemployed (fraction), 25% of the workforce is unemployed (percentage),

0.25 of the workforce is unemployed (decimal)

The same situation can also be expressed as a ratio. A ratio could say that 1 out of 4 of the workforce was unemployed, or alternatively, for every person unemployed, 3 persons were employed.

A special symbol is used for expressing a ratio. The colon sign (:) indicates the important relationship. Thus in terms of the foregoing example, the relationship is 1:4 or 1:3 depending on whether you wish to say:

"1 in every 4 is unemployed" or "For every 1 unemployed, 3 are working".

Another way of expressing a ratio is by using the word "per": 30 kilometres per hour, 60 words per minute, 20 kilometres per day, 2 inches per week, 3 doses per day, 5 meetings per year

Ratios are a particularly important part of the language of business and are used to express important relationships. If you intend to proceed to more advanced accounting studies, a thorough understanding of ratios at this stage will provide a useful foundation for the systematic analysis of accounting information.

To Reduce the Ratio to its Lowest Terms

Put the first figure over the second figure and cancel the resulting fraction. Then re-express as a ratio in the form numerator: denominator, for example:

Calculate the ratio of 17 to 85.

$$\frac{17}{85} = \frac{1}{5}$$

The ratio of 17 to 85 is therefore 1:5 or 1 in 5.

Of course the ratio could be stated as 17:85 but by dividing 85 by 17 it is reduced to the lowest possible terms, and therefore made more manageable. The ratio of 38:171 is better stated as 2:9.

It is important to realize that in its original state, the relationship between the two numbers is not incorrect. But by reducing the terms the relationship becomes a much easier one to follow. This can be seen from the following example:

What is the relationship of 411 to 137? It could be correctly stated as 411:137 but see how much more meaningful the relationship is after the terms have been reduced.

$$\frac{411}{137} = 3$$

The ratio is therefore 3:1.

Divide a Quantity According to a Given Ratio

Add the terms of the ratio to find the total number of parts. Find what fraction each term of the ratio is to the whole. Divide the total quantity into parts according to the fraction. Here again, this is much simpler when actual figures are introduced:

a) RWF60 has to be divided between 2 brothers in the ratio of 1:2. Ascertain the share of each brother.

Total number of parts = 1 + 2 = 3

$\frac{60}{3}$ = RWF20 = one part

Therefore, one brother gets RWF20 (one part), the other RWF40 (two parts).

b) 80 books have to be divided between 4 libraries in the proportions 2:3:5:6. What does each library receive? Total no. of parts = 2 + 3 + 5 + 6 =16

$\frac{80}{16}$ = 5 = one part

Therefore:

library 1 gets 2 x 5 = 10books

library 2 gets 3 x 5 = 15books

library 3 gets 5 x 5 = 25 books and

library 4 gets 6 x 5 = 30 books

80

21.2. Simple interest

Interest (I) is a charge for the use of money for a specific time. This charge is usually expressed as a percentage called the **rate per cent per annum**. Three factors determine the amount of interest:

a) he sum of money on which the interest is payable; this is known as the **principal** (P).

b) The **rate**(R).

c) The length of **time** (Y) for which the money is borrowed.

When the interest due is added to the principal, the sum is called the **amount** (A), which is the amount to be repaid.

Simple interest is interest reckoned on a fixed principal. Simple interest is, therefore, the same for each year, and the total is found by multiplying the interest for one year by the number of years.

Examples

a) Find the simple interest on RWF200 for 3 years at 4% per annum.

Simple interest=$FRw\ 200 * \frac{4}{100} * 3 = Frw\ 24 FRw\ 200 * \frac{4}{100} * 3 = Frw\ 24$

Simple interest=$FRw\ 200 * \frac{4}{100} * \frac{3}{12} = Frw\ 2 FRw\ 200 * \frac{4}{100} * \frac{3}{12} = Frw\ 2$

b) Find the simple interest on RWF200 for 3 months at 4% perannum.To calculate the simple interest on a sum of money lent for a given time at a given rate per cent per annum, the following formula is used:

$$I = \frac{P * R * Y}{100}\ (Y = time\ in\ years)$$

You can use this formula to solve any problem in which we are required to find the principal, rate, time or interest. There are four quantities involved, so given any three; we can find the other one.

When you are working examples always:

State the formula.
Give the value to be substituted.
See that the numbers you use are in the correct units.
Remember to write the correct unit against the answer, not just a number only.

$$R = \frac{I \times 100}{PY}$$

$I = RWF75, \quad P = RWF500, \quad Y = 4$

simple interest will RWF500 earn RWF75 in 4years?

$$R = \frac{75 \times 100}{500 \times 4} = 3\frac{3}{4}\%$$

b) What sum of money will amount to RWF500 in 4 years at 4% per annum simple interest?

$$A = P + \frac{PYR}{100} = P\frac{100 + YR}{100}$$

$$\therefore \quad P = \frac{100A}{100 + YR}$$

$$P = \frac{100 \times 500}{100 + 4 \times 4} = RWF431$$

A=500, Y=4,R=4

21.3. Compound interest

Definition 21.1

In compound interest, the interest due is added to the principal at stated intervals, and interest is reckoned on this increased principal for the next period, and so on, the principal being increased at each period by the amount of interest then due.

Example 21.4

Find the compound interest and the simple interest on RWF1,000 invested at 2 ½ % per annum for 4 years

Therefore, compound interest

= Final amount -Principal

= RWF1,103.81 -RWF1,000

= RWF103.81.

$$1{,}000 \times \frac{2.5}{100} \times 4 = \text{RWF}100$$

b) The simple interest on RWF1,000 at 2 ½ % per annum over 4 years is RWF25 per annum (always constant) or RWF100,

Now work through the above example by yourself to ensure that you fully understand the principle involved, and then answer the following question.

Compound Interest Formula

If P is the principal, and if r is the rate of interest on RWF1 for 1 year, then the interest on P for 1 year is P x r, written as Pr.

At the end of the first year the interest is added to the principal; therefore the new principal = P + Pr or P(1 + r). At the end of the second year the interest on the new principal, i.e. P(1 + r), is P(1 + r) x r or Pr(1 + r). The principal at the end of the second year is now P(1 + r) + Pr(1 + r). This can be written as (P + Pr) (1 + r) which equals $P(1 + r)^2$. You will see that the new principal at the end of n years is equal to $P(1 + r)^n$, and we therefore have the formula for the evaluation of compound interest, which is:

$A = P(1 + r)^n$, where A = Final amount, P = Original sum invested, r = Rate of interest per annum on RWF1 n = Number of years.

Remember that r is the rate of interest on RWF1 for 1 year. Therefore, if the question refers to a rate of interest of 5 **per RWF** per annum,

$$(1 + r) \text{ becomes } 1 + \frac{5}{100} = 1.05.$$

You must become accustomed to thinking in these terms, so that visualising the formula becomes automatic. Learn the formula by heart. Say it to yourself over and over again until it is firmly imprinted on your mind.

Now that you have learned and understood the formula $A = P(1 + r)^n$, it is evident that the real problem lies in the evaluation of $(1 + r)^n$. This is most conveniently done by the use of a calculator or by logarithms, especially when n is large. It is usual to use seven-figure logarithms, as four-figure tables are not sufficiently accurate for compound interest calculations. However, with the time constraints that examinations impose, a calculator is preferable.

Example 21.5

Calculate the compound interest to the nearest cent on RWF1,000 for 2 years at 6% per annum, interest being calculated each six months.

In this case, n is the number of six-month periods (not years) and we must adjust the interest rate accordingly; so 6% p.a. = 3% per half year.

$A = P(1 + r)^n = 1,000\ (1.03)^4$

$= 1,000\ (1.03 \times 1.03 \times 1.03 \times 1.03)$

$= 1,000 \times 1.1255088 = 1,125.50881$

=RWF1,125.51

Therefore, Compound interest = A -P

= RWF1,125.51 - RWF1,000

= RWF125.51 to the nearest RWF.

Having worked carefully through the preceding examples, try to answer the following question.

21.4. Additional Investment

Suppose that you decide to invest RWF2,000 at the beginning of a particular year and that you add RWF100 to this investment at the end of each year. If interest is compounded at 9% per annum, then we can deduce:

The amount invested at the end of the first year is

RWF2,000(1 + 0.09) + RWF100.

The amount invested at the end of the second year is RWF2,000$(1 + 0.09)^2$ + RWF100(1 + 0.09) +100.

The amount invested at the end of the nth year is

RWF2,000$(1 + 0.09)^n$ + RWF100$(1 + 0.09)^{n-1}$ + RWF100$(1 + 0.09)^{n-2}$ + ... + RWF100(1 + 0.09) + RWF100.

Ignoring the first term on the right-hand side, the other terms can be written:

RWF100 + RWF100(1 +0.09)+…+ RWF100$(1 +0.09)^{n-2}$+ RWF100$(1 +0.09)^{n-1}$

This expression is called a geometric progression. The first term is 100 and each successive term is multiplied by (1 + 0.09). This factor of (1 + 0.09) is called the “common ratio”. There is a formula for the sum of such expressions. In this case it is:

$$RWF100\frac{(1.09^n - 1)}{1.09 - 1} = RWF100\frac{(1.09^n - 1)}{0.09}$$

Supposing we wish to know the amount invested after 3 years, then we put n = 3.

$$\text{Amount} = RWF2{,}000(1.09)^3 + RWF100\frac{(1.09^3 - 1)}{0.09}$$

$$= RWF2{,}590.06 + RWF327.81$$

$$= RWF2{,}917.87.$$

Supposing we wish to know the amount invested after 3 years, then we put n = 3.

$$\text{Amount} = RWF2{,}000(1.09)^3 + RWF100\frac{(1.09^3 - 1)}{0.09}$$

$$= RWF2{,}590.06 + RWF327.81$$

$$= RWF2{,}917.87.$$

$$S = P(1+r)^n + a(1+r)^{n-1} + a(1+r)^{n-2} + ... + a(1+r) + a$$

$$= P(1+r)^n + a\frac{(1+r)^n - 1}{(1+r) - 1}$$

$$= P(1+r)^n + a\frac{(1+r)^n - 1}{r}$$

$$S = (P + \frac{a}{r})(1+r)^n - \frac{a}{r}$$

In general, if an amount P is invested at the beginning of a year and a further amount a is invested at the end of each year, then the sum, S, invested after n years is:

We have not attempted to prove that this formula is correct, but have simply stated it. Any proof is outside the scope of this course, but can be found in books, if you are sufficiently interested.

21.5. Introduction to discounted cash flow problems

If a business is to continue earning profit, its management should always be alive to the need to replace or augment fixed assets. This usually involves investing money (capital expenditure) for long periods. The longer the period the greater is the uncertainty and, therefore, the risk involved. With the advent of automation, machinery, equipment and other fixed assets have tended to become more complex and costly. Careful selection of projects has never been so important. One method of selecting the most profitable investments follows.

These techniques do not replace judgement and the other qualities required for making decisions. However, it is true to say that the more information available, the better able a manager is to understand a problem and reach a rational decision.

Classification of Investment Problems

Capital investment problems may be classified into the following types, and each is amenable to discounted cash flow analysis.

- **a)** The replacement of, or improvement in, existing assets by more efficient plant and equipment (often measured by the estimated cost savings).
- **b)** The expansion of business facilities to produce and market new products (measured by the forecast of additional profitability against the proposed capital investment).
- **c)** Decisions regarding the choice between alternatives where there is more than one way of achieving the desired result.
- **d)** Decisions whether to purchase or lease assets.

Basis of the Method

The method is based on the criterion that the total present value of all increments of income from a project should, when calculated at a suitable rate of return on capital, be at least sufficient to cover the total capital cost. It takes account of the fact that the earlier the return the more valuable it is, for it can be invested to earn further income meanwhile.

By deciding on a satisfactory rate of return for a business, this can then be applied to several projects over their total life to see which gives the best present cash value.

For any capital investment to be worthwhile, it must give a return sufficient to cover the initial cost and also a fair income on the investment. The rate which will be regarded as "fair income" will vary with different types of business, but as a general rule it should certainly be higher than could be obtained by an equivalent investment in shares.

Information Required

To make use of DCF we must have accurate information on a number of points. The method can only be as accurate as the information which is supplied.

The following are necessary as a basis for calculation:

- a) Estimated cash expenditure on the capital project.
- b) Estimated cash expenditure over each year.
- c) Estimated receipts each year, including scrap or sale value, if any, at the end of the asset's life.
- d) The life of the asset.
- e) The rate of return expected (in some cases you will be given a figure for "cost of capital" and you can easily use this rate in the same way to see whether the investment is justified).

The **cash flow** each year is the actual amount of cash which the business receives or pays each year in respect of the particular project or asset (a net figure is used). This represents the difference between (c) and (b).

Clearly the receipts and expenditures may occur at irregular intervals throughout the year, but calculations on this basis would be excessively complicated for problems such as may arise in your examination. So, unless you are told otherwise, you can assume that the net receipt or expenditure for the year occurs at the end of the relevant year

Importance of "Present Value"

Before we proceed to a detailed examination of the method used by DCF there is one important concept which you need to understand - the idea of **present value.**

Let us take a businessman who is buying a machine. It will give him, let us say, an output worth RWF100 at the end of the first year, and the same at the end of each successive year. He must bear this in mind when buying the machine which costs, say, RWF1,000. But he must pay out the RWF1,000 now. His income, on the other hand, is not worth its full value now, because it will be a year before he will receive the first RWF100, two years before he will receive the second RWF100, and so on. So if we think of the present value of the income which he is to receive, the first RWF100 is really worth less than RWF100 **now**, and the second RWF100 is worth less still. In fact, the present value of each increment of RWF100 is the sum now which, at compound interest, will represent RWF100 when the sum falls due.
This can easily be calculated, or ascertained from specially prepared **present value tables**, which take account of time and of varying interest rates (see Tables (a)-(d)).

These tables are easily used. We can see, for example, that if we assume a cost of capital of 7%, RWF1 in two years' time is worth RWF0.8734 **now**. This is the sum which would grow to RWF1 in two years at compound interest of 7%. Thus we have established the present value of RWF1 in two years' time, discounted at compound interest of 7%.

We can now look again at the businessman and his machine. We will assume the cost of capital is also 7%. The present value of the first year's income (received at the end of the year, for the purposes of this example) is 100 x RWF0.9346 and the present value of the second year's income is 100 x RWF0.8734. The same method can be used for succeeding years in the same way.

An extract from the present value tables will usually be given with examination questions requiring calculations.

Table 10.1: Present Value of RWF 1 (to 4 sig. figs)

Year \ Rate	2%	3%	4%	5%	6%	7%
1	0.9804	0.9709	0.9615	0.9524	0.9434	0.9346
2	0.9612	0.9426	0.9246	0.9070	0.8900	0.8734
3	0.9423	0.9151	0.8890	0.8638	0.8396	0.8163
4	0.9238	0.8885	0.8548	0.8227	0.7921	0.7629
5	0.9057	0.8626	0.8219	0.7835	0.7473	0.7130
6	0.8880	0.8375	0.7903	0.7462	0.7050	0.6663
7	0.8706	0.8131	0.7599	0.7107	0.6651	0.6227
8	0.8535	0.7894	0.7307	0.6768	0.6274	0.5820
9	0.8368	0.7664	0.7026	0.6446	0.5919	0.5439
10	0.8203	0.7441	0.6756	0.6139	0.5584	0.5083
11	0.8043	0.7224	0.6496	0.5847	0.5268	0.4751
12	0.7885	0.7014	0.6246	0.5568	0.4970	0.4440
13	0.7730	0.6810	0.6006	0.5303	0.4688	0.4150
14	0.7579	0.6611	0.5775	0.5051	0.4473	0.3878
15	0.7430	0.6419	0.5553	0.4810	0.4173	0.3624
16	0.7284	0.6232	0.5339	0.4581	0.3936	0.3387
17	0.7142	0.6050	0.5135	0.4363	0.3714	0.3166
18	0.7002	0.5874	0.4936	0.4155	0.3503	0.2959

Year \ Rate	8%	9%	10%	11%	12%	13%
1	0.9259	0.9174	0.9091	0.9009	0.8929	0.8850
2	0.8573	0.8417	0.8264	0.8116	0.7972	0.7831
3	0.7938	0.7722	0.7513	0.7312	0.7118	0.6931
4	0.7350	0.7084	0.6830	0.6587	0.6355	0.6133
5	0.6806	0.6499	0.6209	0.5935	0.5674	0.5428
6	0.6302	0.5963	0.5645	0.5346	0.5066	0.4803
7	0.5835	0.5470	0.5132	0.4817	0.4523	0.4251
8	0.5403	0.5019	0.4665	0.4339	0.4039	0.3762
9	0.5002	0.4604	0.4241	0.3909	0.3606	0.3329
10	0.4632	0.4224	0.3855	0.3522	0.3220	0.2946
11	0.4289	0.3875	0.3505	0.3173	0.2875	0.2607
12	0.3971	0.3555	0.3186	0.2858	0.2567	0.2307
13	0.3677	0.3262	0.2897	0.2575	0.2292	0.2042
14	0.3405	0.2992	0.2633	0.2320	0.2046	0.1807
15	0.3152	0.2745	0.2394	0.2090	0.1827	0.1599
16	0.2919	0.2519	0.2176	0.1883	0.1631	0.1415
17	0.2703	0.2311	0.1978	0.1696	0.1456	0.1252
18	0.2502	0.2120	0.1799	0.1528	0.1300	0.1108

Table 10.2: Present Value of RWF 1 (to 4 sig. figs) (Contd)

Year \ Rate	14%	15%	16%	17%	18%	19%
1	0.8772	0.8696	0.8621	0.8547	0.8475	0.8403
2	0.7695	0.8417	0.8264	0.8116	0.7972	0.7831
3	0.6750	0.6575	0.6407	0.6244	0.6086	0.5934
4	0.5921	0.5718	0.5523	0.5337	0.5158	0.4987
5	0.5194	0.4972	0.4761	0.4561	0.4371	0.4190
6	0.4556	0.4323	0.4104	0.3898	0.3704	0.3521
7	0.3996	0.3759	0.3538	0.3332	0.3139	0.2959
8	0.3506	0.3269	0.3050	0.2848	0.2660	0.2487
9	0.3075	0.2843	0.2630	0.2434	0.2255	0.2090
10	0.2697	0.2472	0.2267	0.2080	0.1911	0.1756
11	0.2366	0.2149	0.1954	0.1778	0.1619	0.1476
12	0.2076	0.1869	0.1685	0.1520	0.1372	0.1240
13	0.1821	0.1625	0.1452	0.1299	0.1163	0.1042
14	0.1597	0.1413	0.1252	0.1110	0.09855	0.08757
15	0.1401	0.1229	0.1079	0.09489	0.08352	0.07359
16	0.1229	0.1069	0.09304	0.08110	0.07078	0.06184
17	0.1078	0.09293	0.08021	0.06932	0.05998	0.05196
18	0.09456	0.08081	0.06914	0.05925	0.05083	0.04367

Table 10.3: Present Value of RWF 1 (to 4 sig. figs) (Contd)

Year \ Rate	20%	21%	22%	23%	24%	25%
1	0.8333	0.8264	0.8197	0.8130	0.8065	0.8000
2	0.6944	0.6830	0.6719	0.6610	0.6504	0.6400
3	0.5787	0.5645	0.5507	0.5374	0.5245	0.5120
4	0.4823	0.4665	0.4514	0.4369	0.4230	0.4096
5	0.4019	0.3855	0.3700	0.3552	0.3411	0.3277
6	0.3349	0.3186	0.3033	0.2888	0.2751	0.2621
7	0.2791	0.2633	0.2486	0.2348	0.2218	0.2097
8	0.2326	0.2176	0.2038	0.1909	0.1789	0.1678
9	0.1938	0.1799	0.1670	0.1522	0.1443	0.1342
10	0.1615	0.1486	0.1369	0.1262	0.1164	0.1074
11	0.1346	0.1228	0.1122	0.1026	0.09383	0.08590
12	0.1122	0.1015	0.09198	0.08339	0.07567	0.06872
13	0.09346	0.08391	0.07539	0.06780	0.06103	0.05498
14	0.07789	0.06934	0.06180	0.05512	0.04921	0.04398
15	0.06491	0.05731	0.05065	0.04481	0.03969	0.03518
16	0.05409	0.04736	0.04152	0.03643	0.03201	0.02815
17	0.04507	0.03914	0.03403	0.02962	0.02581	0.02252
18	0.03756	0.03235	0.02789	0.02408	0.02082	0.01801

Table 10.4: Present Value of RWF 1 (to 4 sig. figs) (Contd)

Procedure

Since our DCF appraisal will be carried out before the beginning of a project, we shall have to reduce each of the net receipts/expenditures for future years to a present value. This is "discounting" the cash flow, which gives DCF its name, and it is usually done by means of tables, an extract of which you have already seen. You should remember, incidentally, that at the very start of a project the capital expenditure itself may be made, so that at that point there may be a substantial "negative" present value, since money has been paid out and nothing received. If all the present values of the years of the life of the investment (including the original cost) are added together, the result will be the net present value. This is known as the NPV and is a vital factor, because if it is positive it shows that the discounted receipts are greater than expenditures on the project, so that at that rate of interest the project is proving more remunerative than the stated interest rate. The greater the NPV the greater the advantages of investing in the project rather than leaving the money at the stated rate of interest. But if the NPV is a minus quantity, it shows that the project is giving less return than would be obtained by investing the money at that rate of interest.

A practical example will probably be helpful at this point.

Example 21.7

A businessman is considering the purchase of a machine costing RWF1,000, which has a life of 3 years. He calculates that during each year it will provide a net receipt of RWF300; it will also have a final scrap value of RWF200. Alternatively, he could invest his RWF1,000 at 6%. Which course would be more advantageous?

First we must work out the cash flow:

	Receipts	**Payments**	**Net Receipts**
Year 0	Nil	RWF1,000	-RWF1,000
Year 1	RWF300	Nil	+RWF300
Year 2	RWF300	Nil	+RWF300
Year 3	RWF500	Nil	+RWF500

(Remember that the scrap value will count as a receipt at the end of the third year.)

But the businessman could be earning 6% interest instead; so this is the cost of his capital, and we must now discount these figures to find the present value. We can use the extract from the tables which we have already seen.

	Net Receipts	**Discount Factor**	**Present Factor**
Year 0	- RWF1,000	1.0000	- RWF1,000.00
Year 1	+ RWF300	0.9434	+ RWF283.02
Year 2	+ RWF300	0.8900	+ RWF267.00
Year 3	+ RWF500	0.8396	+ RWF419.80
		Net present value	- RWF30.18

As we have seen above, a negative NPV means that the investment is not profitable at that rate of interest. So the businessman would lose by putting his money into the machine. The best advice is for him to invest at 6%.

TWO BASIC DCF METHODS

You have now seen a simple example of how DCF is used, and you already have a basic knowledge of the principles which the technique employs. There are two different ways of using DCF - the yield (or rate of return) method, and the net present value method, which was used in the above example.

The important point to remember is that both these methods give identical results. The difference between them is simply the way they are used in practice, as each provides an easier way of solving its own particular type of problem.

As you will shortly see, the yield method involves a certain amount of trial-and-error calculation. Questions on either type are possible, and you must be able to distinguish between the methods and to decide which is called for in a particular set of circumstances.

In both types of calculation there is the same need for accurate information as to cash flow, which includes the initial cost of a project, its net income or outgoings for each year of its life, and the final scrap value of any machinery.

Yield (Internal Rate of Return) Method

This method is used to find the yield, or rate of return, on a particular investment. By "yield" we mean the percentage of profit per year of its life in relation to the capital employed. In other words, we must allow for **repayment of capital** before we consider income as being profit for this purpose. The profit may vary over the years of the life of a project, and so may the capital employed, so an **average** figure needs to be produced.

DCF, by its very nature, takes all these factors into account.

The primary use of the method is to evaluate a particular investment possibility against a guideline for yield which has been laid down by the company concerned. For example, a company may rule that investment may only be undertaken if a 10% yield is obtainable. We then have to see whether the yield on the desired investment measures up to this criterion. In another case, a company may simply wish to know what rate of return is obtainable from a particular investment; thus, if a rate of 9% is obtainable, and the company's cost of capital is estimated at 7%, it is worth its while to undertake the investment.

What we are trying to find in assessing the figures for a project is the yield which its profits give in relation to its cost. We want to find the exact rate at which it would be breaking even, i.e. the rate at which discounted future cash flow will exactly equal the present cost, giving an NPV of 0. Thus if the rate of return is found to be 8%, this is the rate at which it is equally profitable to undertake the investment or not to undertake it; the NPV is 0. Having found this rate, we know that if the cost of capital is above 8%, the investment will be unprofitable, whereas if it is less than 8%, the investment will show a profit. We thus reach the important conclusion that once we have assembled all the information about a project, the yield, or rate of return, will be the rate which, when used to discount future increments of income, will give an NPV of 0. We shall then know that we have found the correct yield.

You should ensure that you know exactly how and when to use the method, as practical questions are very much more likely than theoretical ones in the examination.

When to Use the Yield Method

This is not a difficult problem, because you will use the method whenever you require to know the rate of return, or yield, which certain increments of income represent on capital employed. You must judge carefully from any DCF question whether this is what you need to know.

How to Use the Yield Method

The calculation is largely dependent on trial and error. When you use this method, you know already that you are trying to find the rate which, when used to discount the various increments of income, will give an NPV of 0. You can do this only by trying out a number of different rates until you hit on the correct result. A positive NPV means that the rate being tried is lower than the real rate; conversely, a negative NPV means that too high a rate is being used. So you need to work the problem out as many times as is necessary to hit on the appropriate rate for obtaining the NPV of 0. If this process is done sensibly, for simple problems such as those which we are going to encounter, it should not take many steps to hit upon the right result. Watch out for any instructions concerning "rounding" of yields - for example, "to the nearest ½ %".

Example 21.8

A businessman is considering investment in a project with a life of 3 years, which will bring a net income in the first, second and third years of RWF800, RWF1,000 and RWF1,200 respectively. The initial cost is RWF2,500 and there will be no rebate from scrap values at the end of the period. He wishes to know, to the nearest 1%, the yield which this would represent. Using the present value tables given earlier, make the necessary calculation.

Year	Net Income/ Outgoings	Discount Factor	Discounted Present Value
0	- RWF2,500	1.0000	- RWF2,500.00
1	+ RWF800	0.9346	+ RWF747.68
2	+ RWF1,000	0.8734	+ RWF873.40
3	+ RWF1,200	0.8163	+ RWF979.56
		Net present value	+ RWF100.64

Table 10.5

We must begin by choosing a possible rate, and testing to see how near this is. Let us try 7%. Referring to the tables, we reach the following results:

Year	Net Income/ Outgoings	Discount Factor	Discounted Present Value
0	- RWF2,500	1.0000	- RWF2,500.00
1	+ RWF800	0.9091	+ RWF727.28
2	+ RWF1,000	0.8264	+ RWF826.40
3	+ RWF1,200	0.7513	+ RWF901.56
		Net present value	- RWF44.76

Table 10.6

A positive NPV, as we have seen, means that we have taken too low a rate for our attempt. Let us try 10% instead:

This time we have obtained a negative NPV so our rate of 10% must be too high. We now know that the rate must be between 7% and 10%. Only a proper calculation can give us the true answer, but having obtained a positive NPV for 7% and a negative NPV for 10%, the approximate rate can be ascertained by interpolation using the formula:

Rate: X= $\frac{a}{a+b}(Y-X)\frac{a}{a+b}(Y-X)$, where

X = Lower rate of interest used

Y = Higher rate of interest used

a = Difference between the present values of the outflow and the inflow at X%

b = Difference between the present values of the outflow and the inflow at Y%

We can extend the trial and error technique as follows. + RWF100 is further from zero than -RWF44 so, 7% is further from zero NPV than 10%. So we shall try 9%.

Clearly, since we are working to the nearest 1% we are not going to get any closer than this. However, if you have time available, there is no reason why you should not check the next nearest rate (in this case, 8%) just to check that you already have the nearest one.

Year	Net Income/ Outgoings	Discount Factor	Discounted Present Value
0	- RWF2,500	1.0000	- RWF2,500.00
1	+ RWF800	0.9174	+ RWF733.92
2	+ RWF1,000	0.8417	+ RWF841.70
3	+ RWF1,200	0.7722	+ RWF926.64
		Net present value	+ RWF2.26

So the yield from this investment would be 9%.

Alternatively, interpolation may be performed graphically rather than by calculation, as shown in Figure 10.1. The discount rate is on the horizontal axis and the net present value on the vertical axis. For each of the two discount rates, 7% and 10%, we plot the corresponding net present value. We join the two points with a ruled line. The net present value is zero where this line crosses the horizontal axis. The discount rate at this point is the required internal rate of return. From Figure 10.1 we see that the rate is 9% correct to the nearest 1%, and this confirms the result of the calculation.

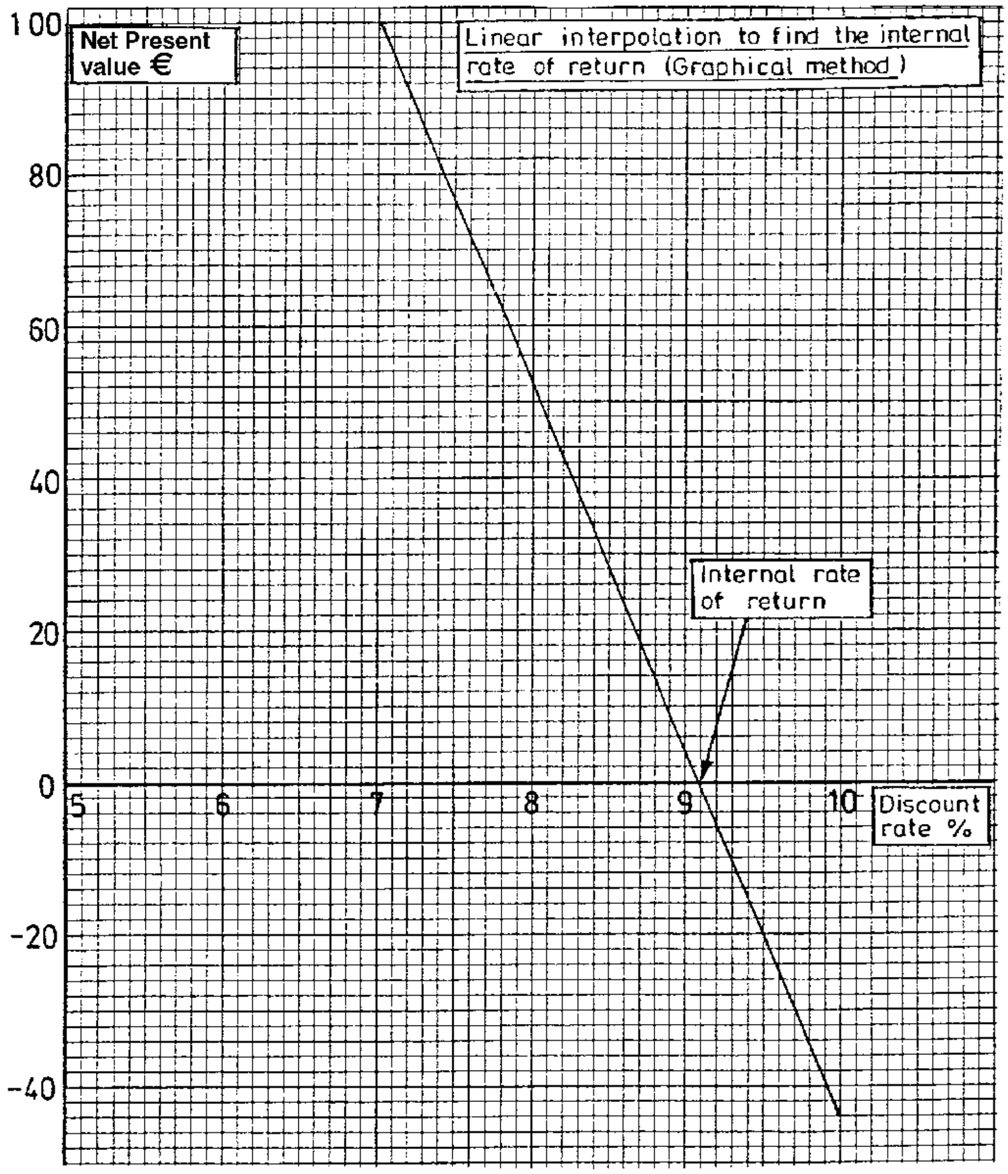

Net Present value €
Linear interpolation to find the internal rate of return (Graphical method)
100
80
60
40
20
0
-20
-40
5
6
7
8
9
10
Internal rate of return
Discount rate %

Figure10.8

Net Present Value (NPV) Method

The NPV method is probably more widely used than the yield method, and its particular value is in comparing two or more possible investments between which a choice must be made. If a company insists on a minimum yield from investments of, say, 10%, we could check each potential project by the yield method to find out whether it measures up to this. But if there are several projects each of which yields above this figure, we still have to find some way of choosing between them if we cannot afford to undertake all of them.

At first sight the obvious choice would be that which offered the highest yield. Unfortunately this would not necessarily be the best choice, because a project with a lower yield might have a much longer life, and so might give a greater profit. However, we can solve the problem in practice by comparing the net present values of projects instead of their yields. The higher the NPV of a project or group of projects, the greater is its value and the profits it will bring. We must remember that in some instances the cost of capital will be higher for one project than for another. For example, a company which manufactures goods may well be able to borrow more cheaply for its normal trade than it could if it decided to take part in some more speculative process. So each project may need to be assessed at a different rate in accordance with its cost of capital. This does not present any particular problems for DCF.

NPV Method and Yield Method Contrasted

You should now be able to see the important difference between the NPV method and the yield method. In the yield method we were trying to find the yield of a project by discovering the rate at which future income must be discounted to obtain a fixed NPV of 0. In the NPV method we already know the discounting rate for each project (it will be the same as the cost of capital) and the factor which we are now trying to find for each project is its NPV. The project with the highest NPV will be the most profitable in the long run, even though its yield may be lower than other projects. So you can see that comparison of projects by NPV may give a different result from comparison by yields. You must decide for each particular problem which method is appropriate for it.
Consider Figure 10.9, which shows the NPV profiles of two competing projects, AA and BB.

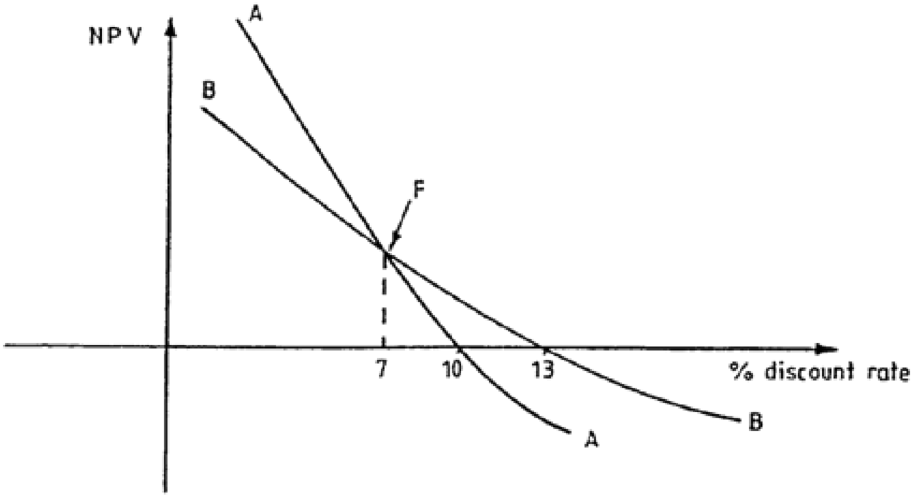


Figure 10.9

From the graph, the yield of project AA is 10% and that of project BB is 13%. The NPV of project AA is greater than that of project BB for discount rates of 0-7%, but at rates greater than 7% the NPV of project BB is greater than that of project AA.

If the company's cost of capital is 7% or less, then project AA will be preferred on an NPV basis, while project BB will be preferred on the basis of the higher yield.

If the company's cost of capital is greater than 7%, project BB will be preferred on both an NPV and yield basis.

The point F, at which the NPV profiles intersect, is called the Fisherian point, after the eminent economist, Irving Fisher. The patterns of cash flows which bring about a Fisherian point can be identified as follows:

a) Where project life-spans vary considerably.
b) Where the cash flows of one project begin at low levels and increase, whilst those of the other begin high and decrease. The discounting process bites more deeply into cash flows in later years because of compounding effects, whereas earlier cash flows are not so severely hit.

How to Use the NPV Method

We must first assemble the cash flow figures for each project. Then, carry out the discounting process on each annual net figure at the appropriate rate for that project, and calculate and compare the NPVs of the projects. As we have seen, that with the highest NPV will be the most profitable.

Example

The ABC Engineering Co. are trying to decide which of the two available types of machine tool to buy. Type A costs RWF10,000 and the net annual income from the first 3 years of its life will be RWF3,000, RWF4,000 and RWF5,000 respectively. At the end of this period it will be worthless except for scrap value of RWF1,000. To buy a Type A tool, the company would need to borrow from a finance group at 9%. Type B will last for three years too, but will give a constant net annual cash inflow of RWF3,000. It costs RWF6,000 but credit can be obtained from its manufacturer at 6% interest. It has no ultimate scrap value.

Type A

Year	Net Cash Income RWF	Discount Factor (9%)	Discounted Present Value RWF
0	- 10,000	1.0000	- 10,000.00
1	+ 3,000	0.9174	+ 2,752.20
2	+ 4,000	0.8417	+ 3,366.80
3	+ 5,000)		
	+ 1,000)	0.7722	+ 4,633.20
		Net present value	+ 752.20

Type B

Year	Net Cash Income RWF	Discount Factor (6%)	Discounted Present Value RWF
0	- 6,000	1.0000	- 6,000.00
1	+ 3,000	0.9434	+ 2,830.20
2	+ 3,000	0.8900	+ 2,670.00
3	+ 3,000	0.8396	+ 2,518.80
		Net present value	+ 2,019.00

able 10.10

Thus we can see that Type B has a far higher NPV and this will be the better investment.

Allowance for Risk and Uncertainty

All investments are subject to risk. In general terms, we mean normal business risk, i.e. **not** that the investment plans will collapse completely as a total write-off, but that unforeseen factors will emerge, such as new legislation, changes in fashion, etc. which make the original estimates of costs and sales, etc. no longer valid. There are two accepted methods for incorporating risk into a capital investment appraisal:

Inclusion of a Risk Premium in the Discount Rate

The inclusion of a risk premium in the discount rate means that if the normal discount rate to be used were, say, 12%, then an additional amount, say 4%, might be allowed to cover for risk, making a 16% discount rate in total. The premium to be added is largely arrived at by subjective rather than objective measurement, and is correspondingly weak. As we have seen also, higher discount rates "bite" more savagely at the more distant cash flows, so that two projects, one short and one long in life-span, would be treated differently for risk by this method.

Attaching Probabilities to Cash Flows

With the first method we effectively looked at the project "normally" - with our usual discount rate and in a "least favourable" position, by requiring the project to provide a higher return to cover risk. We can, in fact, refine this method further by attaching individual probabilities to each cash inflow and outflow, rather than a once-off blanket cover by upping the discount rate.

21.6. Introduction to financial maths

The following are the areas which we cover in this section.

Simple and compound interest, annual percentage rate (APR), depreciation (straight line and reducing balance), discounting, present value and investment appraisal, Annuities, mortgages, amortization, sinking funds.

In this area the letter i and r both stand for the interest rate. The interest rate is often referred to in financial maths as: the discount rate, the cost of capital, the rate of return.

Simple and Compound Interest

If you invest RWF100 in a bank at 10% interest then after one year it will be worth 100(1+10%) =100(1.1) = RWF110.........The 10% is written as a decimal.

The interest here is RWF10.

If this RWF110 is left in the bank another year at 10%, then the simple interest is again RWF10 as in this case no interest is given on the previous interest earned. However, compound interest would be calculated by finding 110(1.1) =121.

The simple interest over 2 years is RWF20. The compound interest is RWF21. The formula to work out the amount in your bank account after n years at r% is Amount $S = P(1+r)^n$

In above example $S = 100(1.1)^2 = 121$.

Annual percentage rate (APR)

In the above example, we assumed interest was added or compounded annually, however sometimes interest may accrue ever six months (twice a year) or over 3 months (4 times per year). (This would of course be better for the customer).

If interest is at 10% per annum we call this the nominal rate, however if it is compounded every six months then the actual return is greater than 10%. We call the rate you are actually getting on your investment the effective rate or actual percentage rate (APR).

Example 21.8

The nominal rate of interest is 10% but interest is being compounded six-monthly. This means that interest is being charged at 5% per six months. Thus RWF100 invested would be worth $100(1.05)^2 = 110.25$ after 1 year so the effective rate is 10.25% and not 10%.

The APR of a nominal rate of 12% compounded quarterly = 12/4 = 3% per quarter $=.03^4$

=12.55%.

Depreciation: Straight line and reducing balance.

Depreciation is an allowance made in estimates, valuations or balance sheets, normally for "wear and tear". There are two techniques for calculating depreciation:

Straight line or equal installment depreciation&

Reducing balance depreciation.

Straight line: if a machine is to depreciate from RWF2500 to RWF500 over 5 years then annual depreciation would be RWF2500-500 = RWF2000/5 =RWF400.

Reducing balance depreciation: remember in compounding we increased an initial investment by $(1+r)^n$, in depreciation we do a similar process in reverse.

For example, RWF2550 depreciated by 15% equals RWF2550(1-0.15) = RWF2550(.85) = RWF2167.50.

Also if RWF2550 was successively depreciated over four time periods by 15% the final depreciated value would be RWF2550$(.85)^4$ = RWF1331.12.

Net Present Value and Internal Rate of Return

This topic describes the technique of present value and how it can be applied to future cash flows in order to find their worth in today's money terms.

If I invest RWF100 in the bank today at 10% annual interest, then after 1 year I would have RWF110.

Looking at this in reverse, if you were due to inherit RWF110 in 1 year, and the interest rate in the bank is 10%, how much is this money worth now. In other words, how much would you need to put in the bank today in order to have RWF110 in one year? Ans: RWF100

The Present Value of RWF110 in one year's time at 10% interest is RWF100.

This is found by taking

RWF110/1.1 = RWF110x1/1.1 = RWF110 x .9090=RWF100

The NPV method of investment appraisal takes into account the “time value of money”. In order to assess an investment where the money earned on the investment is spread over many years the approach taken is to bring all future money amounts back to the present.

Supposing you were given the following investment options; you give me RWF10000 to invest on your behalf. I tell you that I have two different areas where I could invest your money. The return on each is given below:

Year	Option 1	Option 2
1	RWF4000	RWF2000
2	RWF5000	RWF9000
3	RWF4000	RWF2500

Table 10.11

Which option would you choose?

Table 10.12

Year	Option 1	Discount value	Present Value
1	RWF4000	.9090	RWF3636
2	RWF5000	.8264	RWF4132
3	RWF4000	.7513	RWF3005.2
The amount you would need to invest today @ 10% to have the returns indicated in column 1 is the sum of the present values			RWF10773.2

You are receiving these returns and only investing RWF10000 so your Net Present Value is RWF10773.20 - RWF10000 = RWF773.20.

Since the NPV is positive, you must be receiving more than 10% on the investment.

The above problem is usually written as follows:

Table 10.13

Year	Option 1	Discount value	Present Value
0	(RWF10000)	1	(RWF10000)
1	RWF4000	.9090	RWF3636
2	RWF5000	.8264	RWF4132
3	RWF4000	.7513	RWF3005.2
		NPV	**+** RWF773.2

Looking at investment 1 above although with the positive NPV we know that the investment is offering a rate above the discount rate of 10%, we do not know the actual return on the investment. The Internal rate of Return gives us this figure.

What rate of return is the investment yielding?

11%, 12%, 18%??

The rate of return the investment is yielding is called the Internal Rate of Return. If I told you the internal rate of return was 16 % and you found the NPV using 16% what NPV would you expect to get?

The easiest way to find the internal rate of return is to find the NPV using two different discount rates. If the original NPV was positive use a higher rate the second time you discount.

Using option one above we already found the NPV at 10% was RWF773.2. This is positive so we will use a higher discount rate now. You can choose whatever one you want;

Let's use 20%

Table 10.14

Year	Option 1	Discount value	Present Value
0	(RWF10000)	1	(RWF10000)
1	RWF4000	.8333	3333.20
2	RWF5000	.6944	3472
3	RWF4000	.5787	2314.8
		NPV	-880

This is perfect because it is a negative number which is roughly the same as the positive number done earlier

The Internal Rate of Return is then estimated by drawing the following diagram:

Formula: *Internal Rate of Return.*

$N_1 = 773.2$

$r_1 = 10$

$N_2 = -880$

$r_2 = 20$

$$N_1 r_2 - N_2 r_1$$

$$N_1 - N_2$$

$$= \frac{773.2 * 20 - (-880) * 10}{773.2 - (-880} = 14.68$$

Annuities, Mortgages, Amortization, Sinking funds.

This topic deals with various techniques associated with fixed payments (or receipts) over time, otherwise known as annuities.

An annuity is a sequence of fixed equal payments (or receipts) made over uniform time intervals. Some examples are monthly salaries, insurance premiums, mortgage repayments, hire-purchase agreements.

Annuities are used in all areas of business and commerce. Loans are normally repaid with an annuity, investment funds are made up to meet fixed future commitments for example asset replacement, by the payment of an annuity. Perpetual annuities can be purchased with a single lump-sum payment to enhance pensions.

Annuities may be paid

At the end of payment intervals(an ordinary annuity)or
At the beginning of a payment interval (a dueannuity)

There are just 2 formulae you need here:

Accrued amount (compound interest) $A = P(1+i)^n$

Sum of the first n terms of an annuity $S_n = \frac{a((1+r)^n - 1)}{r}$

Table 10.15

This formula is used if an equal amount is lodged over many years.

Amortization of a debt.

If an amount of money is borrowed over a period of time, one way of repaying the debt is by paying an amortization annuity. This consists of a regular annuity in which each payment accounts for both repayment of capital and interest. The debt is said to be amortized if this method is used. Many of the loans issued for houses are like this. This is known as a repayment mortgage.

The standard question is: given the amount borrowed P, with interest of r%, what must the annual payments be A, in order to pay off (amortize) the debt in a certain number of years.

The easiest way to do this is with an "Amortization Schedule".

An amortization schedule is a specification, period by period (normally year by year) of the state of the debt. It is usual to show for each year:

a) Amount of debt outstanding at the beginning of the year.
b) Interest paid
c) Annual payment
d) Amount of principle repaid.

Example:

A debt of RWF5000 with interest of 5% compounded every 6 months is amortized by equal semi-annual payments over the next three years.

a) Find the value of each payment
b) Construct an amortization schedule.

a) Making a standard time period of 6 months, the interest rate is 2.5% with n=6 time periods.

P=5000; n=6; r=0.025 (1+i) =1.025.

We use the formula of Annual discount factors

$$ADF = \frac{1-(1+r)^{-n}}{r}$$

By replacing the value we obtain

$$ADF = \frac{1-(1+0.025)^{-6}}{0.025} = 5.508$$

Thus 5000 = A (5.508)

A= 5000

5.508

A=*RWF*907.7

The amortization schedule is given below:

6 month period	Outstanding debt	Interest paid	Payment made	Principal repaid
1	**5000**	**125**	**907.75**	**782.75**
2	**4217.25**	**105.43**	**907.75**	**802.32**
3	**3414.93**	**85.37**	**907.75**	**822.38**
4	**2592.55**	**64.81**	**907.75**	**842.94**
5	**1749.61**	**43.74**	**907.75**	**864.01**
6	**855.6**	**22.14**	**907.75**	**885.61**
balance	**0.01**			

Sinking fund

Sinking funds are commonly used for the following purposes:

(i) Repayment of debt
(ii) To provide funds to purchase a new asset when the existing asset is fully depreciated.

Dept repayment using a sinking fund:

Here, a debt is incurred over a fixed period of time, subject to a given interest rate. A sinking fund must be set up to mature to the outstanding amount of the debt.

For example: if RWF25000 is borrowed over 3 years at 12% compounded, the value of the outstanding debt at the end of the third year, will be RWF25000 $(1.12)^3$ =RWF35123.20.

If money can be invested at 9.5%, we need to find the value of the annuity, A, which must be paid into the fund in order that it matures to RWF35125.20. Assuming that payments into the fund are in arrears, we need:

$35123.20 = A(1.095)^2 + A(1.095) + A$

$35123.2 = A(3.2940)$

$$A = \frac{35123.2}{3.2940} = 10662.78$$

Or Simply we use the formula

Break-even Analysis

For any business there is a certain level of sales at which there is neither a profit nor a loss, i.e. the total income and the total costs are equal. This point is known as the break-even point. It is very easy to calculate, and it can also be found by drawing a graph called a break- even chart.

Calculation of Break-Even Point – Example

As shown in the last unit, you must be able to layout a marginal cost statement before doing Break Even formulas.

Marginal Cost Statement

Sales	x
- Variable Cost	(x)
= Contribution	x
- Fixed Costs	(x)
= Profit/Loss	xx

Let us assume that the organising committee of a dinner have set the selling price at RWF8.40 per ticket. They have agreed with a firm of caterers that the meal would be supplied at a cost of RWF5.40 per person. The other main items of expense to be considered are the costs of the premises and orchestra which will amount to RWF80 and RWF100 respectively. The variable cost in this example is the cost of catering, and the fixed costs are the amounts for premises and orchestra.

The first step in the calculations is to establish the amount of contribution per ticket.

Contribution

	RWF
Price of ticket(sales value)	8.40
Less Catering cost(marginal cost)	5.40
Contribution	3.00

Now that this has been established, we can evaluate the fixed expenses involved.

Fixed Costs

	RWF
Hire of premises	80
Orchestra fee	100
Total fixed expenses	RWF180

The organisers know that for each ticket they sell, they will obtain a contribution of RWF3 towards the fixed costs of RWF180. Clearly it is only necessary to divide RWF180 by RWF3 to establish the number of contributions which are needed to break even on the function. The break-even point is therefore 60, i.e. if 60 tickets are sold there will be neither a profit nor a loss on the function. Any tickets sold in excess of 60 will provide a profit of RWF3each.

Formulae

$\frac{\text{Fixed costs}}{\text{Contribution per unit}}$ or finding the **break-even** point in volume is:

(this is, of course, exactly what we did in the example).

If the break-even point is required $\text{Break-even point} = \frac{\text{Fixed costs}}{\text{Contribution per unit}}$ **lume**, the formula that should be used is as follows:

The C/s ratio is $\frac{\text{Contribution}}{\text{Sales}} \times 100$.

For example, the contribution earned by selling one unit of Product A at a selling price of RWF10 is RWF4.

C/s ratio= $\frac{RWF\,4}{RWF10}$

× 100 =40%

In our example of the dinner-dance, the break-even point in revenue would be:

rwf 180

3

*rwf*8.40

=RWF504

The committee would know that all costs (both variable and fixed) would be exactly covered by revenue when sales revenue earned equals RWF504. At this point neither profit nor loss would be received.

Suppose the committee were organising the dinner in order to raise money for charity, and they had decided in advance that the function would be cancelled unless at least RWF120 profit would be made. They would obviously want to know how many tickets they would have to sell to achieve this target.

Now, the RWF3 contribution from each ticket has to cover not only the fixed costs of RWF180, but also the desired profit of RWF120, making a total of RWF300. Clearly they will have to sell 100 tickets (RWF300 divided byRWF3).

To state this in general terms:

Volume of sales needed to achieve a given profit **=**

Fixed costs + Desired profit
Contribution per unit

Suppose the committee actually sold 110 tickets. Then they have sold 50 more than the number needed to break even. We say they have a **margin of safety** of 50 units, or of RWF420 (50 × RWF8.40),i.e.

Margin of safety = Sales achieved – Sales needed to break even.

The margin of safety is defined as the excess of normal or actual sales over sales at break- even point. It may be expressed in terms of sales volume or sales revenue.

Margin of safety is very often expressed in percentage terms:

Sales achieved - Sales needed to break even Sales achieved × 100%

1.e. the dinner committee have a percentage margin of safety of 50/110 × 100% =45%.

The significance of margin of safety is that it indicates the amount by which sales could fall before a firm would cease to make a profit. Thus, if a firm expects to sell 2,000 units, and calculates that this would give it a margin of safety of 10%, then it will still make a profit if its sales are at least 1,800 units (2,000 – 10% of 2,000), but if its forecasts are more than 10% out, then it will make a loss.

The profit for a given level of output is given by the formula: (Output × Contribution per unit) – Fixed costs.

It should not, however, be necessary for you to memorise this formula, since when you have understood the basic principles of marginal costing, you should be able to work out the profit from first principles.

Consider again our example of the dinner. What would be the profit if they sold (a) 200 tickets (b) RWF840 worth of tickets?

a) We already know that the contribution per ticket isRWF3.

Therefore, if they sell 200 tickets, total contribution is 200 × RWF3 = RWF600.

Out of this, the fixed costs of RWF180 must be covered: anything remaining is profit.

Therefore profit = RWF420. (Check: 200 tickets is 140 more than the number needed to break even. The first 60 tickets sold cover the fixed costs; the remaining 140 show a profit of RWF3 per unit. Therefore profit = 140 × RWF3 = RWF420, asbefore.)

b) RWF840 worth of tickets is 100 tickets, since they are RWF8.40each.

	RWF
Total contribution on 100tickets=	300
Lessfixed costs	180
Profit	RWF120

Break-even Chart

Information Required

a) Sales Revenue

When we are drawing a break-even chart for a single product, it is a simple matter to calculate the total sales revenue which would be received at various outputs.

As an example let us take the following figures:

Output (units)	**Sales revenue** (RWF)
0	0
2,500	10,000

5,000	20,000
7,500	30,000
10,000	40,000

b) Fixed Costs

We must establish which elements of cost are fixed in nature. The fixed element of any semi-variable costs must also be taken into account.

Let us assume that the fixed expenses total RWF8,000.

c) Variable Costs

The variable elements of cost must be assessed at varying levels of output.

Output (units)	**Variable costs** (RWF)
0	0
2,500	5,000
5,000	10,000
7,500	15,000
10,000	20,000

d) Plotting the Graph

The graph is drawn with level of output (or sales value) represented along the horizontal axis and costs/ revenues up the vertical axis. The following are the stages in the construction of the graph:

a) Plot the sales line from the above figures.

b) Plot the fixed expenses line. This line will be parallel to the horizontal axis.

c) Plot the total expenses line. This is done by adding the fixed expenses of RWF8,000 to each of the variable costs above.

d) The break-even point (often abbreviated to BEP) is represented by the meeting of the sales revenue line and the total cost line. If a vertical line is drawn from this point to meet the horizontal axis, the break-even point in terms of units of output will be found.

The graph is illustrated in Figure 10.16

Note that, although we have information available for four levels of output besides zero, one level is sufficient to draw the chart, provided we can assume that sales and costs will lie on straight lines. We can plot the single revenue point and join it to the origin (the point where there is no output and therefore no revenue). We can plot the single cost point and join it to the point where output is zero and total cost = fixed cost.

In this case, the break-even point is at 4,000 units, or a revenue of RWF16,000 (sales are at RWF4 per unit).

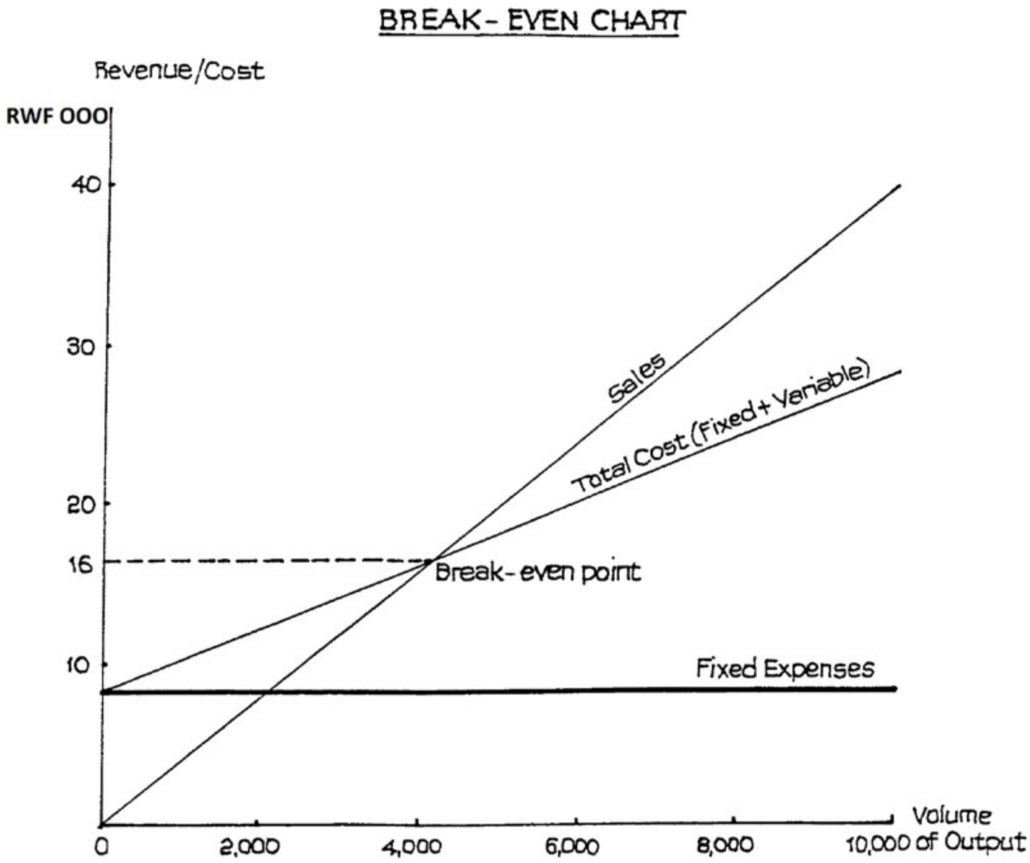


Figure 10.16

Break-even Chart for More Than One Product

Because we were looking at one product only in the above example, we were able to plot "volume of output" and straight lines were obtained for both sales revenue and costs. If we wish to take into account more than one product, it is necessary to plot "level of activity" instead of volume of output. This would be expressed as a percentage of the normal level of activity, and would take into account the mix of products at different levels of activity.

Even so, the break-even chart is not a very satisfactory form of presentation when we are concerned with more than one product: a better graph, the profit-volume graph, is discussed in the next study unit. The problem with the break-even chart is that we should find that, because of the different mixes of products at the different activity levels, the points plotted for sales revenue and variable costs would not lie on a straight line.

Fixed, Variable and Marginal Costs

Introduction

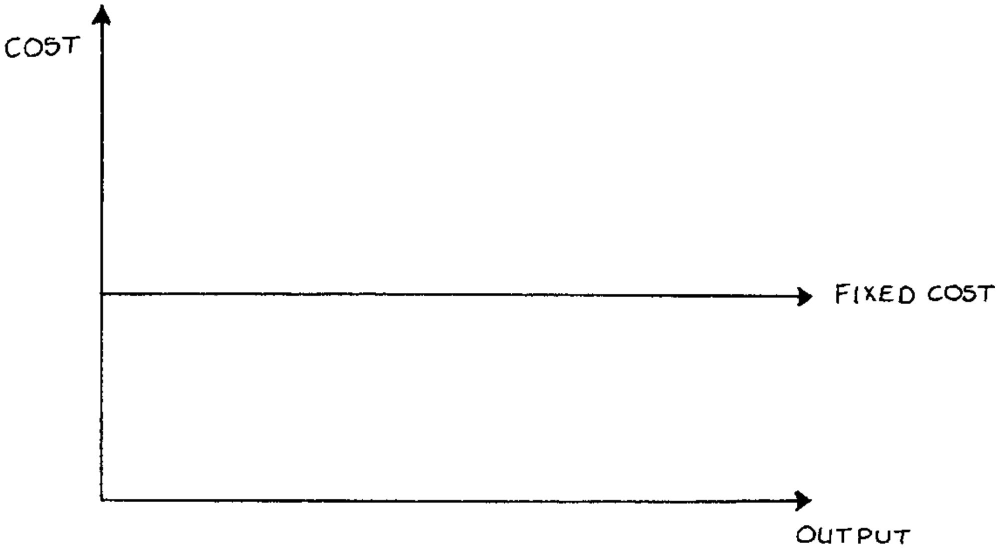
Costs can be divided either into direct and indirect costs, or variable and fixed costs.

Direct costs are **variable**, that is the total cost varies in direct proportion to output. If, for instance, it requires RWF 10 worth of material to make one item it will require RWF20 worth to make two items and RWF100 worth to make ten items and so on. Overhead costs, however, may be either fixed, variable or semi-variable.

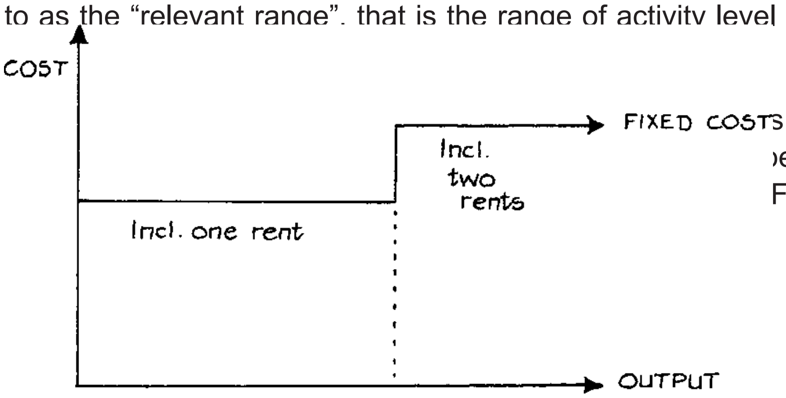
Fixed Cost

A fixed cost is one which **can** vary with the passage of time but, **within limits**, tends to remain fixed irrespective of the variations in the level of output. All fixed costs are overhead. **Examples of fixed overhead are: executive salaries, rent, rates and depreciation.**

A graph showing the relationship of total fixed cost to output appears in Figure10.4.



Please note the words "within limits" in the above description of fixed costs. Sometimes this is referred to as the "relevant range", that is the range of activity level within which fixed costs (and variable costs)



the same no matter what the output of the
e that a second factory is required and a large
Figure10.5.

A cost with this type of graph is known as a step function cost for obvious reasons.

Variable Cost

This is a cost which tends to follow (in the short term) the level of activity in a business. As already stated, direct costs are by their nature variable. **Examples of variable overhead are: repairs and maintenance of machinery; electric power used in the factory; consumable stores used in the factory.**

The graph of a variable cost is shown in Figure 10.6.

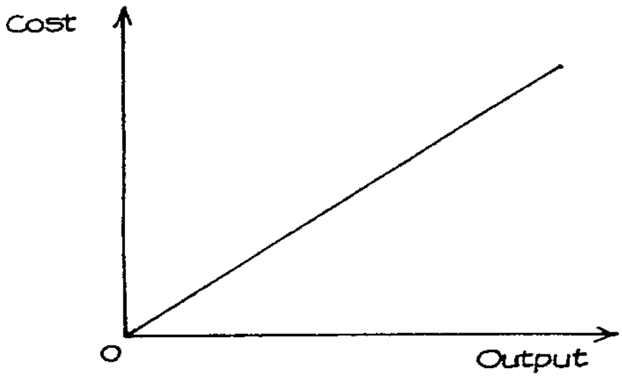


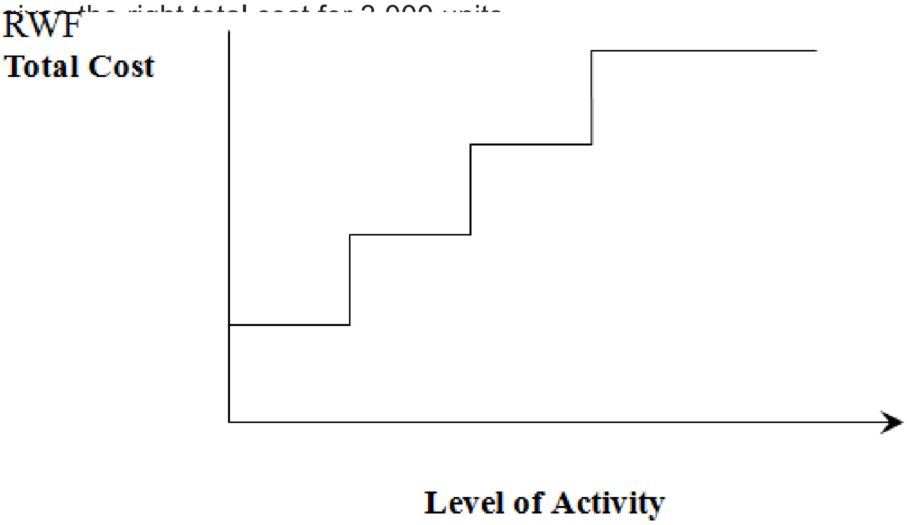
Figure 10.19

Semi-Variable (or Semi-Fixed) Cost

This is a cost containing both fixed and variable elements, and which is thus partly affected by fluctuations in the level of activity. For examination purposes, semi-variable costs usually have to be separated into their fixed and variable components. This can be done if data is given for two different levels of output.

Example

At output 2,000 units, costs are RWF12, 000. At output 3,000 units, costs are RW Therefore for an extra 1,000 units of output, an extra RWF5, 000 costs have been incurred. This is entirely a variable cost, so the variable component of cost is RWF5 per unit. Therefore at the 2,000 units level, the total variable cost will be RWF10, 000. Since the total cost at this level is RWF12, 000, the fixed component must be RWF 2,000. You can check that a fixed component of RWF 2,000 and a variable component of RWF5 per unit gives the right total cost for 3,000 units.



Example

Rent can be a step cost in certain situations where accommodation requirements increase as output levels get higher.

A Step Cost

Many items of cost are a fixed cost in nature within certain levels of activity.

Semi-Variable Costs

This is a cost containing both fixed and variable components and which is thus partly affected by fluctuations in the level of activity (CIMA official DFN).

Example

Running a Car

Fixed Cost is Road Tax and insurance.
Variable cost is petrol, repairs, oil, tyres-all of these depend on the number of miles travelled throughout the year.

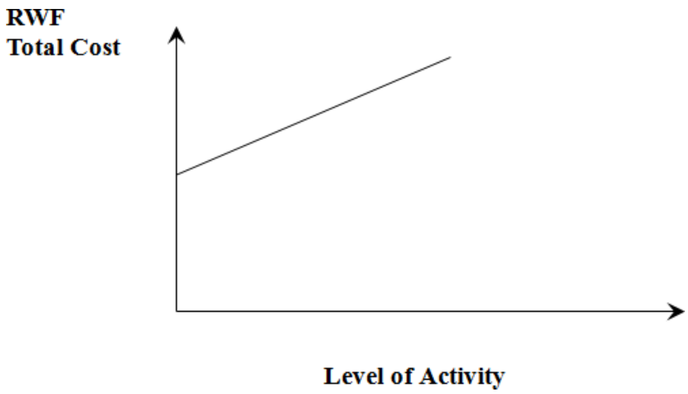


Figure 10.21

A method of splitting semi-variable costs is the High – Low method.

High – Low method

Firstly, examine records of cost from previous period. Then pick a period with the highest activity level and the period with the lowest level of activity.

Total Cost of high activity level minus total cost of low activity level will equal variable cost of difference in activity levels.

Fixed Costs are determined by substitution

CHAP 22.

REGRESSION ANALYSIS

22.1. Correlation

22.1.1. Study objectives

By the end of this chapter, you should be able to:

identify direction of correlation;

find correlation coefficient by spearman rank correlation; and

Compute correlation coefficient by Pearson product moment of correlation.

Definition 22.1: Correlation is a measure of how related two or more variables are. Correlation depends on data that already exist and determines if and in what way is two sets of data related to each other. The purpose of doing correlations is to allow us to make a prediction about one variable based on what we know about another variable.

Correlation is a linear relationship between two random variables.

For example, there is a correlation between number of working hours and sales made. We find that people with longer working hours tend to earn more sales. It is possible to make a prediction if we know there is a correlation between two variables. If we know a firm's profit, we can predict their hours of work**.**

22.1.2. Types of a correlation

The types of correlation include: i) a linear, non-linear and no correlation. In this book however we shall only consider a linear and zero correlation only. There are three patterns that correlations can follow. These are called positive correlation, negative correlation and zero correlation. In a positive correlation, as the values of one of the variables increase, the values of the second variable also increase. The example on sales made and working hours is **a** positive correlation. People with longer working hours tend to make more sales. People with shorter working hours tend to make less sales.

Here are some examples of positive correlations:

The amount of fuel consumed by a vehicle and the distance covered. The more fuel consumed, the longer the distance covered.

The acreage cultivated and the yields on a firm.

In a negative correlation, as the values of one of the variables increase, the values -*' the second variable decrease. Likewise, as the value of one of the variables decreases the value of the other variable increases. Here are some examples of negative correlation: The price of a particular commodity and its demand. As the price increases the demand decreases. When the price for, say, tomatoes go down people tend to buy more. When the price for cement goes up fewer people will tend to buy cement.

22.1. 3. The strength of a correlation

- Correlation, whether positive or negative, vary in their strength from weak to **strong.**

- Positive correlations will be reported as a number between 0 and 1. A score **of 0** means that there is no correlation (the weakest measure). A score of 1 is a **perfect positive** correlation, which indicates total agreement between the two variables, the correlation score gets closer to 1, it is getting stronger. So, a correlation **of** Ois stronger than 0.5; also 0.4 is stronger than0.2.

- Negative correlations will be reported as a number between 0 and -1. Again, a **0** means no correlation at all. A score of -1 is a perfect negative correlation, which indicates total disagreement between the two variables. As the correlation **score** gets close to -1, it is getting stronger negatively. So, a correlation of -0.8 is **stronger** than -0.6.

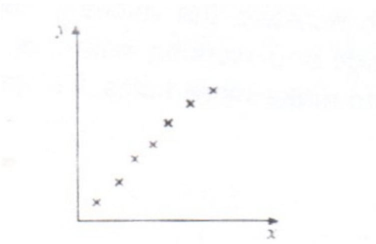
- Remember that the negative sign does not indicate anything about strength. It is a symbol to tell you that the correlation is negative in direction. When judging the strength of a correlation, just look at the modulus of the number and ignore the sign**.**

Correlation between two sets of data can be demonstrated on scatter diagrams.

22.1.4. Scatter diagrams

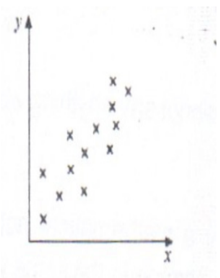
The diagrams below show the different scatter diagrams and the different types of correlation that exist between data value.

Case 1



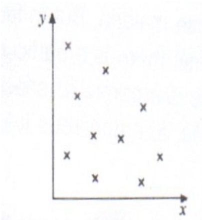
This case shows a strong positive correlation between x and y. The points lie close to a straight line with y increasing as x increases.

Case 2.



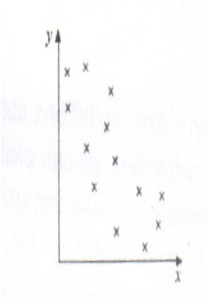
This case shows a weak, positive correlation between x and y. The trend shown is that y increases as x increases but the points are not close to a straight line.

Case 3



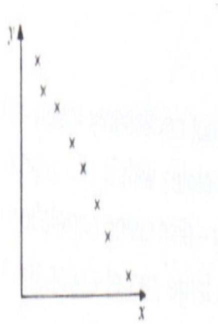
In the case above, we see no correlation between x and y; the points are distribute: randomly on the graph.

Case 4



From the above diagram, shows a weak, negative correlation between x and y. The trend shown is that y decreases as x increases but the points do not lie close to a straight line

Case 5



The diagram above shows a strong, negative correlation. The points lie close to a straight line, with y decreasing as x increases.

Note: If the points plotted were all on a straight line we would have perfect correlation, but it could be positive or negative as shown in the diagrams above.

A correlation tells us that the two variables are related, but we cannot say anything about whether one caused the other. "Correlation does not imply causation". Causation isn't really a mathematical notion. Correlation means that as one value changes, another variable change in the same way. Causation means that when one value changes, it causes the other to change. There is a very big difference between causation and correlation. To give a rather commonly abused example: the majority of malaria cases are diagnosed between the months of June and July. That's also the same period of time when there are lots of mangoes and maize. So, people see the correlation between eating lots of mangoes and maize and the diagnosis of malaria, and assume that mangoes and maize cause malaria. But in fact, there is no causal linkage. The causal factor in both cases is time: there is a particular season when both mangoes and maize are abundant; and there is a particular season when mosquitoes, the insects responsible for transmitting malaria, are numerous It just happens that these seasons coincide.

To show causation, you need to show a mechanism for the cause, and demonstrate that mechanism experimentally. So, when someone shows you a correlation, what you should do is look for a plausible causal mechanism, and see if there's any experimental data to support it. Without a demonstrable causal mechanism, you can't be sure that there's a causal relationship - it's just a correlation.

Advantages of correlation

One major advantage of the correlation method is that we can make predictions about things when we know about correlations. If two variables are correlated, we car predict one based on the other. For example, we know that mock examination scores and UNEB results are positively correlated. So. when teachers want to predict who is likely to succeed at their schools, they will choose students with high mock examination scores.

Disadvantages of correlation

The biggest disadvantage of correlation is that many times people tend to forget that correlation is not a measure of the cause. As earlier stated, correlation is not causation yet the two are very often confused. Scientists are careful to point out that correlation does not necessarily mean causation. The assumption that A causes B simply because A correlates with B is a logical fallacy. However, sometimes people commit the opposite fallacy - dismissing correlation entirely-as if it does not imply causation. This would dismiss a large part of important scientific evidence

So $r_s = \frac{6\sum d^2}{n(n^2-1)} r_s = \frac{6\sum d^2}{n(n^2-1)}$

$= 1 - \frac{6(38)}{8(64-1)} = 1 - \frac{6(38)}{8(64-1)}$ =0.5476

Tied ranks

Sometimes two or more readings may have the same value. In such cases, we proceed as follows:
Suppose we have the heights y (in cm) of five men166, 168, 165,168, 170. When this happens the two equal readings 168 and 168 are given the same rank:
170 would be ranked 1 or 1st, but the two 168's which would have been ranked 2 and 3; because they are equal, they are both ranked 2.5(the arithmetic mean of 2 and 3). The next rank is then 4. The complete ranking is as follows:

Height y	166	168	165	168	170
Rank of y	4	2.5	5	2.5	1

Sometimes three readings may have the same value for example166,165.170,172, 165, 175, 165,168 would be ranked as follows:

Height y	166	165	170	165	172	165	175	168
Rank of y	8	6	3	6	2	6	1	4

All the three equal numbers have been given the same rank. If the three 165's were not equal they would be ranked 5, 6 and 7. Because they are equal they were all ranked 6 (the arithmetic means of 5, 6 and 7) and the next rank is 8.

Example 2

The table shows the points earned by eight students in contest x and contest y.

Contest, x	64	65	70	65	80	75	85	85
Contest, y	58	54	58	54	65	58	62	73

Required:

Rank the results and find the value of Spearman's coefficient of rank correlation. Comment on your results.

Solution:

X	6 4 -	65	-70	65	80	75	85	85	
Rank x	8	6.5	5x	6.5	3	4	1.5	1.5	
Y	58	54	58	54	65	58	62	73	
Rank y	5	7.5	5	7.5 ‘	2	5	3 ‘	1	
Rand d	3	-1	0	-1	1	-1	-1.5	0.5	
d^2	9	1	0...	1	1	1	2.25	0.25	$\sum d^2 = 145$

So $r_s = \frac{6\sum d^2}{n(n^2-1)} r_s = \frac{6\sum d^2}{n(n^2-1)}$

$= 1 - \frac{6(14.5)}{8(64-1)} = 0.8297$

Merits of Spearman's Rank Correlation

i) Since in this method$\sum d \sum d$ or the sum of the differences between R_1. and R_2 is always equal to zero, it provides a check on the calculation.

ii) Since Spearman's Rank Correlation is the same thing as Karl Pearson's Coefficient of Correlation between ranks, it can be interpreted in the same way as Karl Pearson's Coefficient of correlation.

iii) Rank correlation unlike Karl Pearson's Coefficient of Correlation does not assume normality in the universe from which the sample has been take-

iv) Rank Correlation is very easy to understand and apply.' However, Pearson's Coefficient is based on a set of full information while Spearman's Coefficient s based only on the ranks. The values of obtained by these two methods would generally differ.

v) Spearman's Rank method is the only way of studying correlation between qualitative data which cannot be measured n figures but can be arranged -serial order.

Pearson product moment for the coefficient of correlation

Pearson product moment correlation coefficient, r is given by the formula

$r = \frac{n(\sum xy)-(\sum x)(\sum y)}{\sqrt{[n(\sum x^2)-(\sum x)^2)(n(\sum y^2)-(\sum y)^2)}}$ $r = \frac{n(\sum xy)-(\sum x)(\sum y)}{\sqrt{[n(\sum x^2)-(\sum x)^2)(n(\sum y^2)-(\sum y)^2)}}$ where n is the number of pairs.

Example 3

The time heavy weight lifters use to practice and the amount of milk each lifter consumes per session was evaluated at the end of the week and the results are shown in the table below.

Lifter	H1	H2	H3	H4	H5	H6	H7	H8	H9

Hours	3	0	2	5	8	5	10	2	
Quantity of milk	48	8	32	64	10	32	56	72	48

Required:

i) Compute the Pearson product moment correlation coefficient for the above data.

ii) Comment on the value obtained.

Solution:

i)

Lifter	Hours(x)	Milk(y)	xy	X^2	V^2
H1	3	48	144	9	2304
H2	0	8	0	0	64
H3	2	32	64	4	1024
H4	5	64	320	25	4096
H5	8	10	80	64	100
H6	5	32	160	25	1024
H7	10	56	560	100	3136
H8	2	72	144	4	5184
H9	1	48	48	1	2304
	$\sum x = 36$	$\sum y = 370$	$\sum xy = 1520$	$\sum x^2 \sum x^2$=232	$\sum y^2 \quad \sum y^2 =$ 19236

Pearson product moment correlation coefficient $r = \frac{n(\sum xy)-(\sum x)(\sum y)}{\sqrt{[n(\sum x^2)-(\sum x)^2)(n(\sum y^2)-(\sum y)^2)}}$

$r = \frac{n(\sum xy)-(\sum x)(\sum y)}{\sqrt{[n(\sum x^2)-(\sum x)^2)(n(\sum y^2)-(\sum y)^2)}}$

where n = 9 is the number of pairs.

r = $\frac{9(1520)-(36)(3700}{\sqrt{[9(232)-362][9(19236)-(370)^2]}} = .0687)3\,d.p \frac{9(1520)-(36)(3700}{\sqrt{[9(232)-362][9(19236)-(370)^2]}} = .0687)3\,d.p$

vi) The computed value of r = 0.067, shows a very weak positive correlation between the two variables.

Merits of Pearson's Coefficient of Correlation

i) Pearson's Coefficient of Correlation not only gives an idea about the covariation of the two series but also indicates the direction of relationship. Thus, this coefficient measures both the degree and direction of the correlation between two variables.

ii) It is the most widely used method of studying relationship between inter-related phenomena. .

Demerits of Pearson's Coefficient of Correlation

i) It assumes a linear relationship between the variables even though it may not be there.

ii) It is liable to be misinterpreted, as a high degree of correlation Hp does not necessarily mean very close relationship between the variables.

iii) It is tedious to calculate.

iv) It is unduly affected by the values of extreme items. For example, if the values of —series are 1, 2, 3, 4, 5 and the corresponding values of y—series are 5. 4, 3, 2, 1, the coefficient of correlation between them would be zero. However, if we add one more value in the -series and its value is 100 and add one more value in the y series and its value is 120, the coefficient of correlation of these six pairs of values would be +.99. Thus, the addition of a single set of large figures has boosted up the correlation from zero to +.99.

22.1.5. Coefficient of determination

The coefficient of determination is the amount of variation that is explained by the coefficient of determination is the amount of variation that is explained by the regression line. It is computed as $r^2 = \frac{Explained\ variation}{total\ variation}$ $\frac{Explained\ variation}{total\ variation}$. We can simply square the linear coefficient of correlation. Example if r = 0.897, the coefficient of determination would be r2 = (0.879)2= 0.805. From which we can state that 80.5 % of the total variation of one variable is explained by the other variable, the other 19.5 % is attributed to other factors.

Self-test questions

Question 1

A trainer of accountants wanted to compare two teaching methods A and B. He selected a random sample of accountant trainees and grouped them in two pairs so that the trainees in a pair have approximately equal scores in a basic accounts test. In each pair one student was i taught using method A and the other using method B and were all examined after the course. The marks obtained by the trainees are tabulated below.

Pair	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
A	24	29	19	14	30	19	27	30	20	28	11
B	37	35	16	26	23	27	19	20	16	11	21

Required:

Find the correlation coefficient between the two sets of marks by rank order method.

Solution:

The correlation coefficient is 0.8181.

Question 2

The ranks of 15 students in two subjects, economics and accountancy, are given below. The two values within brackets represent the rank of the same student in economics and accountancy, respectively.

(1,10), (2,7). (3,2), (4,6), (6,8), (7,3), (8,1), (9,11), (10,15), (11,0), (12,5), (12,5). (13,14), (14,12): and (15,13).

Required:

Use Spearman's method to find the correlation coefficient between the performance in economics and accountancy.

Solution:

The correlation efficient is 0.5143.

Question 3

A sales manager wishes to establish if there is a correlation between the number advertisements in the New Vision daily paper per week and the amount of sales at the factory in million Rwandan Francs .

No. of adverts	2	5	8	8	10	12
Sales in million Frw	2	4	7	6	9	10

Required:

a) Use the above data and calculate the Pearson product moment correlation coefficient.

b) Comment on the correlation. *

Solution:

a) The correlation efficient is 0.9878.
b) The correlation is strong.

Question 4

The following table shows output data for tested machines at a factory and output data for machines which were put to use without testing them.

Output of tested (tons)	480	490	510	510	530	550	610	640
Output of untested(tons)	270	290	330	290	310	300	320	370

Required:
Determine the coefficient of correlation.
Solution:
The correlation coefficient is 0.8117 and a moderately strong correlation.
Question 5

The table below shows ages and weights often members in a health club:

Ages	56	42	72	36	**63**	42	68	60	47	55
Weight (kg)	147	125	160	118	!149	140	152	155	128	150

Calculate the rank correlation coefficient for the data and comment on the results.
Solution: The correlation coefficient is 0.7121 and a moderately positive correlation.
Question 6

The following data are marks awarded to CPA candidates in two papers I and II

Paper I	57	58	59	59	60	61	62	64
Paper II	77	78	75	78	82	82	79	81

Use the method of rank correlation to determine the relationship in performance between the two papers. Comment on the relationship

Solution:
The rank correlation coefficient by Spearman's formula is 0.6905 which implies a high, positive correlation.
Question 7

Applicants for a job in a company were interviewed by two of the personnel staff. Each of eight applicants was awarded by each of the interviewers. The marks are given in the table below.

	A	B	C	D	E	F	G	H
Interviewer 1	22	27	24	17	20	22	16	13
Interviewer 2	28	23	25	14	26	17	20	15

Required:
Calculate to two decimal points, the spearman's rank correlation between the two sets of marks.

Solution: The Spearman's rank correlation is 0.59.

Question 8

A given set of data resulted into the following.
$\sum x = 680$, $\sum y = 996$, $\sum x^2 = 20154$, $\sum y^2 = 34670$, $\sum xy = 24844$ and $n = 30$.

Required:
Use the data to calculate Pearson product moment correlation coefficient.
Solution:
The Pearson product moment correlation efficient is 0.8228.

Question 9
The table shows the advertising expenditure (x) and sales revenue (y)

Advertising exp. (million Frw)	1	2	3	4	5
Sales revenue (million Frw)	1	1	2	2	4

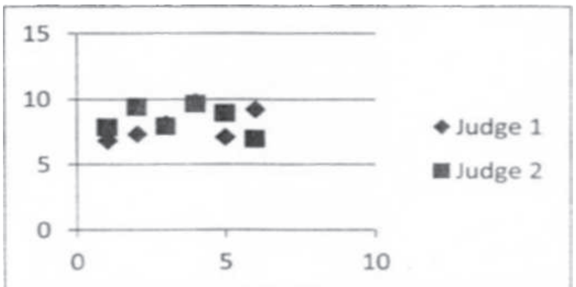
Required:

Find the correlation coefficient and the coefficient of determination for the sample data listed above and interpret your results.
Solution: The correct answer's 0.95, $r^2 = 0.9025$; thus, 90.25%-variation in sales is due to advertising.

Question 10

The table below shows the marks awarded by two judges to six students who participated in a financial reporting competition.

Student	A	B	C	D	E	F
Judge 1	6.8	7.3	8.1	9.8	7.1	9.2
Judge 2	7.8	9.4	7.9	9.6	8.9	6.9



a) Draw a scatter plot.
b) Calculate Spearman coefficient of rank correlation and comment on your results.

a) **Solution:**
b)The Spearman coefficient of rank correlation is 0.2571. thus, a weak correlation.

Question 11
The table shows a set of measurements for eight students on internship at max certified auditor's firm.

Name	Height (cm)	Reading speed	Shoe size	Ring **size**
Aguti	172	200	12	3
Akiria	170	160	10	6
Abamu	168	170	11	7 5
Abayo	170	150	8	6 5
Aine	166	150	8	5
Asimwe	172	185	11	7
Adutia	166	135	7	6
Akello	169	165	9	7.5

Required:

a) Calculate the coefficient of correlation for each pair.
b) Calculate the coefficient of determination for each pair.

Solution:

a) Pearson's correlation for each pair:
  i. Height and reading speed $\mathbf{r_{xy}}$= 0.80
  ii. Height and shoe size r_{xy}= 0.79
  iii. Height and ring size r_{xy}= 0.65
  iv. Reading speed and shoe size $\mathbf{r_{xy}}$ **=** 0.94
  v. Reading speed and ring sizer_{vv} = 0.87
  vi. Shoe size and ring size $\mathbf{r_{xy}}$ **=** 0.79

b) Coefficient of determination for each pair:
  i. Height and reading speed = 0.64
  ii. Height and shoe size = 0.58
  iii. Height and ring size = 0.42
  iv. Reading speed and shoe size = 0.88
  v. Reading speed and ring size = 0.76
  vi. Shoe size and ring size = 0.62

22.2. Regression

22.2.1. Study objectives

By the end of this chapter, you should be able to:

- explain the individual terms used in the simple linear regression model;
- determine the least squares regression equation; and
- make point estimates for the dependent variable.

22.2.2. Introduction

A line of best fit drawn by eye is dependent upon the subjective judgment of the person who draws it. Consequently, the position of the line will differ slightly from person tc person. A line of best fit independent of individual judgment will have to be drawn mathematically. Such a line is called a regression line.

22.2.3. Regression

Regression is a way of analysing relationship between one dependent variable (response) and one or more independent variables (predictors).

22.2.4. Distinction between correlation and regression

Correlation	Regression
i) There is no distinction between the dependent and explanatory variables.	1. There is a distinction between dependent variable and explanatory variables.
ii) The relationship is between two variables.	2. Predicts a particular value of y for a specific value of x
iii) Measures the strength or degree of linear relationship between two variables.	3.Estimates the mean value of a dependent variable for given values Df x

Uses of regression line

- Used in estimating the mean or average value of the dependent variable given the values of the independent variables.
- Used in predicting or forecasting the mean value of the depended: variable give-the values of the independent variables beyond the sample range

22.2.5. Forms of regression

There are two:

Simple linear regression and Multiple linear regression

Simple linear regression

A simple, linear regression is a regression equation in which we have one independent variable and one outcome variable.

Multiple linear regression

A multiple linear regression is an equation in which we have multiple independent variables with one outcome variable.

Variables in regression

Independent variable

Defined as that variable controlled by the investigator or defined as that variable which explains another variable. It is the explanatory variable, normally denoted by x.

Dependent variable

Defined as the explained variable or the outcome variable. It is denoted by Y.

Methods of determining a regression line

Drawing a line of best fit

When drawing a line of best fit, an attempt is made to minimise the total divergence of points from the line. The number of points left on either side of the line drawn is approximately equal.

The general assumed regression line is y = a + bx, where a and b are constants. Describing the line of best fit means finding the appropriate values of a and b. The b is the gradient of the line of best fit. It describes the rate of change of y with respect to x. The a is the value of y when x is zero. Once drawn, a line of best fit shows the estimated relationship between x and y and any value of x is used to obtain the prediction for y.

Least squares approach

When drawing a line of best fit, an attempt is made to minimise the total divergence of points from the line. In computing the line mathematically, the same idea is pursued. This approach is logically known as the method of least squares.

Regression of y on x

It is the regression line in which deviations are measured vertically. In order to find a and b in the equation y =a + bx, it is necessary to solve the two simultaneous equations

∑y=an+b∑x...(1)

∑xy=a∑x+b∑$x^2 x^2$

....(2)

Where n= the number of pairs of figures:

The least squares method gets b as $b=\frac{n(\sum xy)-(\sum x \sum y)}{n(\sum x^2)-(\sum x)^2} b=\frac{n(\sum xy)-(\sum x \sum y)}{n(\sum x^2)-(\sum x)^2}$

Example 1

For the data set below:

No.	1	2	3	4	5	6	7	8	9	10	11	12
X	65	50	55	65	55	70	65	70	55	70	50	55
y	85	74	76	90	85	87	94	98	81	91	76	74

Required:

Describe the simple regression line**.**

Solution:

$\sum(x)=725 \sum(y)=1011 \sum(xy)=61685 \sum(x^2)=44{,}475\ \bar{x}=60.417.\bar{y}=84.25.$

$n = 12$
$b = \frac{(12x61685) - (725x1011)}{(12x44.475)}$

=0.897

A = 84.25-$(0.897)(60.417)(0.897)(60.417)$
=30.056

The regression equation is y = 30.056 + 0.897x. Interpretation of results:

a) 0.056 is the value of y when x=0 or point where the regression line cuts the y-axis.
b) 0.897 is the rate of change in y per unit change in x.

The simple linear regression equation y = 30.056 + 0.897x can be used to predict any value of y for any given value of x. For example, y(52) = 76.7 and y(75) = 97.331

Regression of x on y

If it is desired Jo compute the second regression line, known as the regression line of x on y, it is only necessary to alter the two simultaneous equations so that the x's and y's are interchanged. The equations therefore become:

$\sum x = an + b\sum y \sum x = an + b\sum y$...**(3)**

$\sum xy = a\sum y + b\sum y^2 \sum xy = a\sum y + b\sum y^2$...**(4)**

The least squares method gets b as $b = \frac{n(\sum y)-(\sum yx\sum y^2)}{n(\sum y^2)-(\sum x\ )} b = \frac{n(\sum y)-(\sum yx\sum y^2)}{n(\sum y^2)-(\sum x\ )}$ and a=$\bar{x}\bar{x}$-b$\bar{y}\bar{y}$

Example 2
Taking the values in the table below.
Required:
To describe the line x = a +by of the given data points:

X	2	5	4	6	3
y	60	100	70	90	80

Solution:

X	Y	xy	y^2
2	60	120	3600
5	100	500	10000
4	70	280	4900
6	90	540	8100
3	80	240	6400
$\sum x = 20$	$\sum y = 400$	$\sum xy = 1{,}680$	$\sum y^2 = 33{,}000$

The least squares method obtains$b = \frac{n(\sum y)-(\sum x\sum y)}{n(\sum y^2)-(\sum y)^2} b = \frac{n(\sum y)-(\sum x\sum y)}{n(\sum y^2)-(\sum y)^2}$

$b = \frac{5(1980)-(20)(400)}{5(3300)-(400)^2} b = \frac{5(1980)-(20)(400)}{5(3300)-(400)^2} = 0.08. \bar{x} = \frac{20}{5} \bar{x} = \frac{20}{5} = 4\ \bar{y}\bar{y} = \frac{400400}{5\ \ 5} = 80$

a = 4-(0.80)(80)=-2.4

Therefore, the equation of the line is x = - 2.4 + 0.08y.

Interpretation of results:

i. 2.4 is the value of x when y=0 or point where the regression line cuts the x- axis.

ii. 0.08 is the rate of change in x per unit change in y.

The simple linear regression equation x = -2.4 + 0.08ycah be used to predict any value of x for any given value of y. For example, x(65) = 2.8 and x(120) = 7.2.

Self-test questions

Question 1

Briefly distinguish between the terms ‘line of best fit’ and ‘line of regression’. Peak Talk Telephone Company Ltd has increased the number of its- retail shops. This ha; led to increased sales as indicated in the following table.

Number of retail shops (x)	Sales (Frw billion) (y)
900	0.75
1000	0.90
1100	0.95
1300	1.05
1350	1.20

Given that y = mx + b gives the line of best fit

$$m = \frac{n(\sum xy) - (\sum x \sum y)}{n(\sum x^2) - (\sum xy^2)}$$

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{[n(\sum x^2) - [n(\sum xy^2) - (\sum y)^2]}}$$

Where r is the coefficient of correlation,

Required:

i) Determine the values of m and b and hence find the equation of the line of best fit.

ii) If Peak Talk Telephone Company Ltd opens 50 more new retail shops, estimate the total sales.

iii) Determine the value of r and interpret the results

Solution:

i) m = 0.00052, b= 16.95, y = 0.00052x + 16.95

ii) 1.01

iii) r = 1.7152

Question 2

To obtain a mini statement (MS) and to use the Automatic Teller Machine(ATM), a customer $_0$f Stanbic Bank (U)'Ltd (S) and Barclays Bank (U) Ltd (B) is charged user fees, in Rwandan Francs , as shown below:

Bank	Charge on MS	ATM
S	6,000	680
B	8,000	400

On a certain day, 15 and 12 customers, respectively, obtained mini statements from their banks and used the ATM.

Required:

Determine the amount of money collected by each bank from the above customers.

Solution:

Bank S got 15 x 680 = 10,200

Bank B got 400 x 12 = 4,800

Question 3

The following summary of statistics were recorded from set a of data of the number of hours (x) a student was absent and the final score (y%) obtained in each subject:$n = 7 \sum x\ x\ 5$ $n = 7 \sum x\ x\ 5$
$\sum y = 511$ $\sum y = 511$, $\sum x^2 = 579,\ \sum y^2 = 38{,}993, \sum xp = 3{,}745 \sum x^2 = 579,\ \sum y^2 = 38{,}993, \sum xp = 3{,}745$

Required:

i) Determine the coefficient of correlation and comment on the relationship between hours: absent and performance.

ii) Find the equation of regression for the data

Hint $r = \frac{n(\sum xy)-(\sum x)\sum y}{\sqrt{(n\sum x^2-(\sum x)^2)(n\sum y^2-(\sum y)^2)}} r = \frac{n(\sum xy)-(\sum x)\sum y}{\sqrt{(n\sum x^2-(\sum x)^2)(n\sum y^2-(\sum y)^2)}}$,
$b = \frac{n(\sum xy)-(\sum x)\sum y}{(n\sum x^2-(\sum x)^2)} b = \frac{n(\sum xy)-(\sum x)\sum y}{(n\sum x^2-(\sum x)^2)}$ and $a = \frac{\sum y - b\sum x}{n} a = \frac{\sum y - b\sum x}{n}$

Solution:

i) Y= 4926.5 +1.68x

ii)5010.5

Question 4

A company keeps records of its monthly expenditure for advertising and its total monthly sales. For the first 10 months in 2010, the records showed the following.

Advertising cost (Frw million)	43	44	36	38	47	40	41	54	37	46
Sales (Frw million)	74	76	60	68	79	70	71	94	65	78

Required:

Find the least squares regression equation for the data. Use the regression equation in c) i) above to predict the sales if the company plans to spend Frw 50 million for advertising in the following month, assuming that other factors can be neglected.

Solution:

y = 112 + 2.15*

x = -28.84

Question 5

Distinguish between regression and correlation.

The training officer of Xerox Consult extracted the following data from a trainee's notebook:

ariable x	17	11	8	13	5	7	9	14	10	6
Variable v	45	39	30	42	18	26	32	43	35	20

∑x= 100, ∑y = 330. ∑x²=1130, ∑y² =11708, ∑xy =3612

Required:

i) Calculate the coefficient of correlation between x and y variables and interpret the result.

ii) Calculate the values of a and b in the regression equation y = a + bx.

iii) Plot the x and y values and the regression line on the same axes.

Solution:

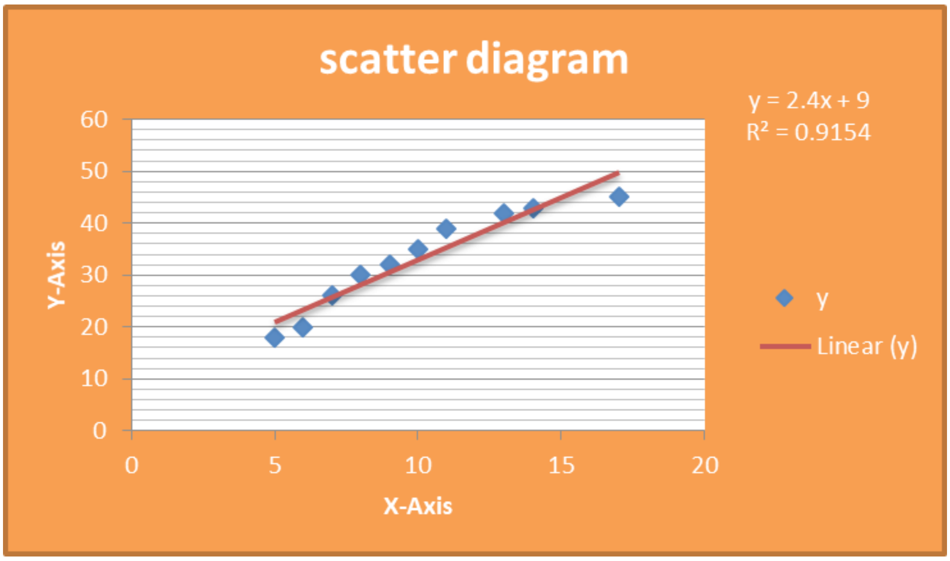
$$r = \frac{n(\sum xy) - (\sum x)\sum y}{\sqrt{(n\sum x^2 - (\sum x)^2)(n\sum y^2 - (\sum y)^2)}} = \frac{103612 - 100x330}{\sqrt{(10x1130 - (100)^2)(10x11708 - (330)^2)}} = \frac{3210}{3263.4}$$

$$= 0.9561$$

The two variables are highly correlated, ii)

a = 9 and b = 2.4.

X	17	11	8	13	5	7	9	14	10	6
y	45	39	30	42	18	26	32	43	35	20



Question 6

a) A first order model is defined by the equation y =$\beta_0\beta_0$ +x$\beta_1\beta_1$ x passes through the points (3,4) and (0,2).

Required:

i. Find the equation of the model.

ii. State the slope.

b) An appliance store conducts a 5 - month experiment to determine the effect of advertising on sales revenue. The results are:

Advertising expenditure x (million FRW)	Sales revenue y (million FRW)
1	1
2	1
3	2
4	2
5	4

Required:

i) Plot sales revenue against advertising expenditure.

ii) Draw a line of best fit and determine its equation.

iii) Estimate the sales revenue when advertising is Frw 4.5 million.

Solution:

a) i) $\mathbf{y = \beta_o + \beta_1 x}$

Substituting gives 4= $\beta_0\beta_0$ +3$\beta_1\beta_1$.................................1

2=$\beta_0\beta_0$ + 0...2

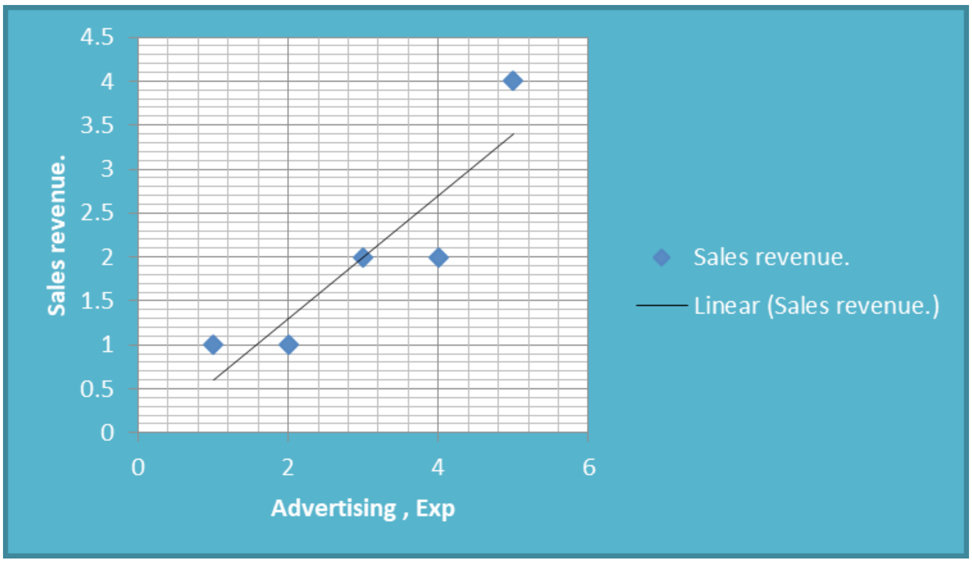
$\beta_0\beta_0$=1 $\beta_1\beta_1$=1

Therefore, equation is **y** = 1 + **x**

ii)Slope is 1 (coefficient of x).

a) **i)**

Advertising , Exp.x)	**1**	**2**	**3**	**4**	**5**
Sales revenue.	**1**	**1**	**2**	**2**	**4**



ii) line of best fit y = **x -** 1.
ii) When **x =** 4.5 y = Frw 3.5 million.

Question 7

The table shows the number of departments **(x)**each with the number of staff members (y):

x	2	3	4	5	6	7
y	2	3	3	4	6	6

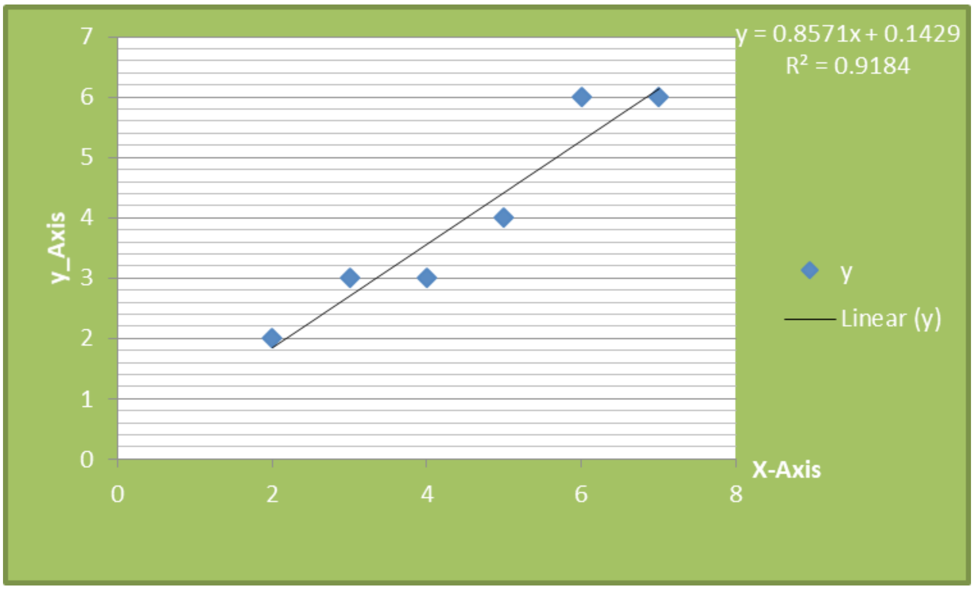
REQUIRED:

i) Find the least squares estimate of ($\beta_0\beta_0$ and $\beta_1\beta_1$.
ii) Plot the data points and draw the least squares line.
iii) Does the line pass through the data points?

Solution :

i)

x	2	3	4	5	6	7
y	2	3	3	4	6	6



iii)Yes.

Question 8

A farmer is interested in investigating the relationship between the yield of potatoes (y and the amount of the new fertilizer (x) that is applied to the potato plants. He divided a field into eight plots of equal size and applied differing amounts of fertiliser to each The yield of potatoes (in kg) and the fertiliser application (in kg) were recorded for each plot. The data are shown in the table below.

x	1	1.5	2	2.5	3	3.5	4	4.5
Y	26	31	27	28	36	35	32	34

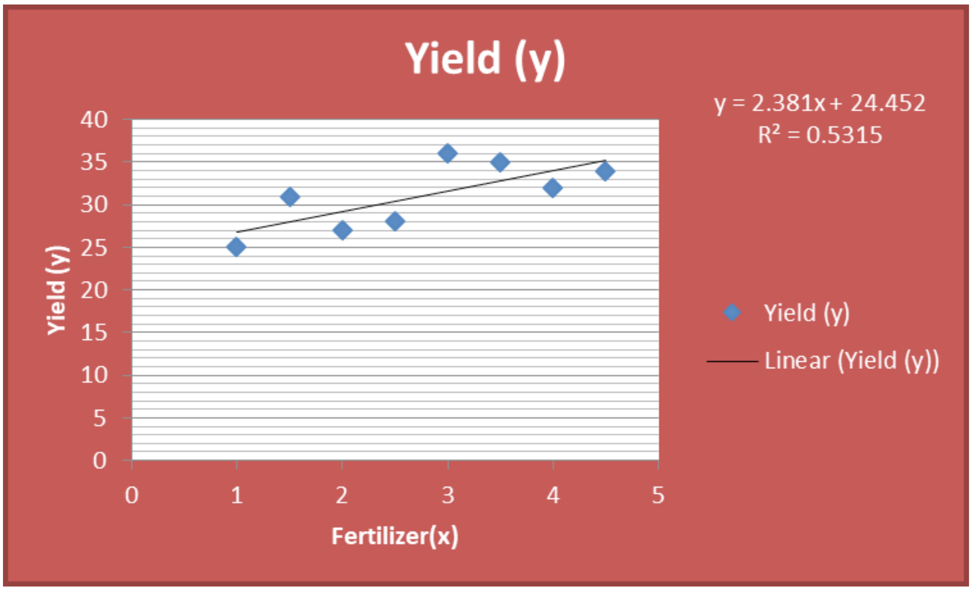
Required:

i) Construct-a scatter diagram for the data.
ii) Find the least squares estimates for $\beta_0\beta_0$ and $\beta_1\beta_1$.
iii) According to the least squares line, estimate how many kg of potatoes can be obtained from a plot to

which 3.75 kg of fertiliser had been applied.

Solution:

Fertilizer(x)	1	1.5	2	2.5	3	3.5	4	4.5
Yield (y)	25	31	27	28	36	35	32	34



ii)y = 1.142x + 22.4

$\beta_0\beta_0$ = 22.4 and $\beta_1\beta_1$= 1.142

iv) when x = 3.75 y = 26.6825 kg

Question 9

The relationship between a company's^ sales and the total sales for a particular industry given in the table.

Week	Company sales y (million Frw)	Industry sales x (million Frw)
1	5	100
2	10	120
3	10	130
4	14	150
5	13	140
6	16	150

Use the data in the table to find the least squares line that describes the relationship between a company's sales and the total sales for a particular industry.

Solution:

Week	Company sales y (million Frw)	Industry sales x (million Frw)	xy	x^2
1	5	100	500	10000
2	10	120	1200	14400
3	10	130	1300	16900
4	14	150	2100	22500
5	13	140	1820	19600

6	16	150	2400	22500
Sum	68	790	9320	105900

$y = a + bx$

$b = \frac{n(\sum xy)-(\sum x)\sum y}{(n\sum x^2-(\sum x)^2)} b = \frac{n(\sum xy)-(\sum x)\sum y}{(n\sum x^2-(\sum x)^2)}$ and $a = \frac{\sum y - b\sum x}{n} a = \frac{\sum y - b\sum x}{n}$

$$b = \frac{6 * 9320 - (790)(68)}{(6 * 105900 - 790^2)} = 0.195\ and\ a = \frac{68 - 0.195 * 790}{6} = -14.4$$

$y = -14.4 + 0.195x$

TIME SERIES ANALYSIS FORECASTING

23.1. Study objectives

By the end of this chapter, you should be able to:

- understand the purposes of forecasting; be able to calculate a moving average;
- understand the principles of exponential smoothing;
- be able to analyse or decompose a time series; and understand how to use regression equation in forecasting.

23.2. Definition of concepts

a) **Time series**

Time series is an arrangement of statistical data in accordance with the time of its occurrence. When we record numerical observations of an individual variable that is a function of time at different points of time, the sets of observations obtained constitute a time series.

b) **Forecasting**

Forecasting is an attempt to predict the feature by using qualitative or quantitative means. Most data in business and economic publications constitute time series. The annual production of coffee in Rwanda over the last 10 years. The monthly sales of a supermarket for the last 2 years. The imports of a country over a period of six years.

The analysis of time series data usually focus on two types of problems: Attempting to estimate the factors (components) that produce the patterns in the time series. Using the estimates of time series in forecasting future behaviour of the series.

23.3. Importance of time series analysis

It helps in the understanding of the past behaviour. The past trend helps in predicting the future behaviour. It enables us to predict or forecast the behaviour of the phenomenon in future which is very essential for business planning.
It helps in making comparative studies in the values of different phenomenon at different times or places.
It helps in the evaluation of current achievements.

- The segregation and study of various components of time series is of paramount importance to a businessman in the planning of future operations and policy decisions.
- The main objective of analysing time series is to understand, interpret and evaluate changes in economic phenomena in the hope of more correctly anticipating the course of future events.

23.4. Components of time series

The main components of time series are:

a) Secular fend

The general tendency of the time series to increase or decrease or stagnate during a long period of time is called **secular trend** or **simply trend**. This phenomenon is usually observed in most of the series that show growth. For example, an **upward** tendency is usually observed in time series relating to population, production and sales of products, prices, income, bank deposits, etc. while a downward tendency is notice d in the time series relating to death, epidemics etc. due to advancement in technology, improved medical facilities etc. Secular trend is regular, smooth and long term movement of a statistical series. It reveals the general tendency of the data.

b) Season variation

It represents a periodic movement where the period is no longer than one year. The factors which mainly cause this type of variation in time series are climatic changes of the different seasons and the customs and habits which people follow at different times. The short-range stock and brisk periods of business activity at different seasons of the year, production and consumption of commodities, sales and profits of a company, etc. are in fact attributed to seasonal variations.

The main objective of the measurement of seasonal variations is to isolate them from the attend and study their effects. A study of the seasonal patterns is extremely useful to- businessmen, producers, sale-managers etc. It helps in planning future operations and formulation of policy, in decisions regarding purchase or production, inventory control, personal requirements, selling and advertising.

c) Cyclical variations

Cyclical variations or fluctuations are another type of periodic movement, with **a** period more than one year. Such movements are fairly regular in nature. One complain period is called a cycle**.** Cyclical fluctuations are found to exist in most of the business and economic time series. The four phases of business cycle are usually x uncompleted over a period of 8 to 10 years. These phases are: (i) **prosperity,** (ii) decline, (iii) **depression** and (iv) **recovery.**

d) Irregular variation

Irregular variations or ‘movements are -such variations which are completely unpredictable in character. These are unforeseen variations usually caused by factors which are either wholly unaccountable or caused by such unforeseen events as war, flood strikes and lockouts etc. These may sometimes be the result of many small forces each of which has a negligible effect but their combined effect is not negligible. They are in most cases beyond human control.

4) Seasonal adjustment

Seasonal adjustment is the process of estimating and then removing from a time series influences that are systematic and calendar related. Observed data needs to be seasonally adjusted as seasonal effects can conceal both the true underlying movement in the series, as well as certain non-seasonal characteristics which may be of interest to analysts.

5) **Deflating time series**

It is the overall average level of say prices. Deflation is the opposite of inflation

6) **Methods of calculating trends**

a) **Moving averages method**

In this method, instead of taking the actual values relating to the particular years we take into account the moving averages of the values of three, four, five or more years. In order to compute a three-year moving average, the values against the first, second, third years are averaged. This average value is written against middle year that is against the second year. Similarly, the values against the second, third and fourth year are averaged and average value is written against the 3rd year and so on. If it is required to compute 4-year moving average, then central moving average is computed. These moving average values are plotted on the same graph paper on which the time series have been drawn. This would smooth out irregularities eliminating short-term changes and will show long - time tendency.

Example 1

The table below shows the production of maize in tonnes per year for nine years**.**

Year	Quantity
2000	20
2001	32
2002	52
2003	30
2004	40
2005	22
2006	70
2007	34
2008	23

Required:

From the data above compute:

i) a three year - moving totals; and

ii)a three year – moving average.

Solution:

i)The first three moving total is calculated thus; 20 + 32 + 52 = 104 which is entered opposite year 2002. The next is calculated 32 + 52 + 30 = 114, and so on.

ii)The first three moving average is calculated thus $\frac{20+30+52}{3} = \frac{104}{3} = 34.7\frac{20+30+52}{3} = \frac{104}{3} = 34.7$ which is entered opposite year 2001. The next is calculated $\frac{32+52+30}{3} = \frac{134}{3} = 38\frac{32+52+30}{3} = \frac{134}{3} = 38$ and so on.

Year	3 - moving total	3 - moving average
2000		
2001		34.7
2002	104	38
2003	114	40.7
2004	122	30.7
2005	92	46.7
2006	140	34.7

2007	104	31
2008	93	

Merits of moving averages

- This method is simple as compared to the method of least squares.
- It is a flexible method of measuring trend. If a few more figures are added to the data, the entire calculation are not changed, we only get some more trend values.
- If the period of moving averages happens to coincide with the period of cyclical fluctuations in the data, such fluctuations are automatically eliminated
- The moving average has the advantage that it follows the general mc cements of the data and that its shape is determined by the data rather than the statistician's choice of a mathematical function.

Limitations

- Trend values cannot be computed for all the years. The longer the period of moving averages the greater the number of years for which trend values cannot be obtained. For example, in the three-year moving average, trend values cannot be obtained for the first year and last year, in five-yearly moving average for the firs: two years and the last two years, and so on. It is often these extreme years in which we are most interested.

- Great care has to be exercised in selecting the period of moving average. No hard and fast rules are available for the choice of the period and one has to use his own judgment.
- Since the moving average is not represented by a mathematical function, this method cannot be used in forecasting which the main objective of trend analysis is.
- Although theoretically we say that if the period of moving average happens to coincide with the period of cycle, the cyclical fluctuations are completely eliminated but in practice since the cycles are by no means perfectly periodic, the lengths of the various cycles in any given series will usually vary considerably and therefore no moving average can completely remove the cycle.

d) Method of least squares

The method of least squares is an appropriate type of mathematical model for obtaining the trend represented by an equation. This is determined on the basis of the given data by the principle of least squares. This principle states that **the constants should be chosen so as to make the sum of the squares of the errors of estimate the minimum possible**. Method of least squares is a mathematical technique used to fit **a** trend line for a given data with the following two conditions: *

$z(y - y_0) = o$ i.e. the sum of the deviations of the actual values of Y and the compound values of Y is zero.

The sum of squares of the above deviations $\sum(y - \mathbf{Yg})\mathbf{2}\sum(y - \mathbf{Yg})\mathbf{2}$is least where Y is the observed time series value and k_0 is the corresponding trend value given by the trend line. The line obtained by this method is the line of best fit.

The equation of the straight line is of the form y = **a** + bx

In this equation, there are two constants **a** and b to be determined from the normal] equations which are:

$$\sum y\sum y = \mathbf{na + bx}, \sum \mathbf{xy}\sum \mathbf{xy} = a \sum x\sum x + \mathbf{b} \sum \mathbf{x2}\sum \mathbf{x2}$$

Solving these normal equations simultaneously we get the values of the unknown and b. Substituting these numerical values of the unknown a and b in the equation a straight line we get the required equation which represents the straight line be fitted to the data.

Merits of least squares

This is a mathematical method of measuring trend and as such there is possibility of subjectiveness. The line obtained by this method is called the line of best fit because it is this! line from where the sum of the positive and negative deviations is zero and that sum of the squares of the deviations is least, that is, $\Sigma(\mathbf{y}-y0)\Sigma(\mathbf{y}-y0)$ = o and $\Sigma(y - \mathbf{y0})2\Sigma(y - \mathbf{y0})2$ least.

Limitation

Mathematical curves are useful to describe the general movement of a time series but it is doubtful whether any analytical significance should be attached to them, except in special cases.

Significance of trend values

The object of finding the time series trend is to enable the underlying tendency. The data to be highlighted. Thus, a business sales trend will normally show whether are moving up or down (or remaining statue in the long term.

Techniques for extracting the trend

There are three techniques that can be used to extract a trend from a set c time series values

a) Semi-averages. This is the simplest technique, involving the calculation of two (x,y) averages which', when plotted on a chart as two separate points and joined up, form a straight line.
b) Least squares regression,. It is a mathematical technique used to fit a trend line for a given data. The equation of the straight line is of the form y = a + bx
c) Moving averages. This is the most commonly used method for identifying a trend and involves the calculation of a set of averages. The trend, when obtained and charted, consists of straight line segments.

Estimation of production/output using a trend line

Using a trend line got by least squares regression, the trend can be as in the example below.

Example 2

The data below is for the passengers moving out of a park by taxi in different years:

	Passengers moving out by taxi							
	Year 1				Year 2			
Quarter	1	2	3	4	1	2	3	4
No of passengers ('000)	22	50	59	72	52	60	54	80

Calculate the regression line.

Solution:

Let **x** = time point representing quarters numbered from 1 to 8. And **y** = number of passengers corresponding to each quarter.

X	y	xy	X^2
1	22	22	1
2	50	100	4
3	59	177	9
4	72	288	16
5	52	260	25
6	60	360	36
7	54	378	47
8	80	640	64
36	449	2.225	204

From the table $\sum x$ **= 36;** $\sum y$ **= 449 ;** $\sum xy$**= 2225;** $\sum X^2$ **= 204 :** n=8

Putting the regression line as **y** = a + bx, we find a and b

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \sum(x)^2} b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \sum(x)^2} = \frac{8x2225 - 36x4498x2225 - 36x449}{8x204 - 36^2 \quad 8x204 - 36^2}$$

$$= \frac{17{,}800 - 16{,}16417{,}800 - 16{,}164}{1632 - 1296 \quad 1632 - 1296} = \frac{1632}{336} = 4.9(1dp)\frac{1632}{336} = 4.9(1dp)$$

Hence, the regression line for the trend is t = 34.1 + 4.9x.

(**Note**: Once the regression line is determined, it will be used for calculating trend values. So, the normal 'y' is replaced by 't'.)

The time point values (x= 1, 2, 3 etc) can now be substituted into the above regression line to give the trend values required.

Graphical representation of data

In graphical representation of data, we are given a set of data which is to be plotted on a graph and used to estimate any other outcome in future. After plotting a line of best fit is drawn, it leaves almost the same number of points on either side. The following example shows how data is presented graphical.

Example 3

The table below shows ages and heights of 10 members forming a village council

Age(Years)	53	39	69	33	60	39	65	57	44	52
Height (m)	1.47	1.25	1.60	1.18	1.49	1.40	1.52	1 55	1.28	1.50

Required:

Represent the above data on a graph and fit a line of best fit.

Solution

Question 6

At K.K Boxing Club, the weights (x)kg and the height (y) cm of the members were recorded as shown in the table below:

X(kg)	Y (cm)
55	150
59	153
65	155
70	157
75	159
80	165
81	168

Find the equation of the regression line of y on x in the form y = a + bx.

Solution:

a = 115.707 b = 0.61248

Question 7

Calculate the 4 and 5 monthly averages and totals of the following data.

Month	Sales
Jan	900
Feb	980
Mar	1010
Apr	970
May	890
Jun	990
Jul	1110
Aug	1060
Sept	1130
Oct	980
Nov	1110
Dec	1090

Solution

Month	Sales	4MT	4MA .	5MT	5MA
Jan	900				
Feb	980				
Mar	1010		965	-	950
Apl	970	3860	962.5		968
May	890	3850	965	4750	994
Jun	990	3860	990	4840	1004
Jul	1110	3960	1012.5	4970	1036
Aug	1060	4050	1072.5	5020	1054

Sept	1130	4290	1070	5180	1078
Oct	980	4280	1070	5270	1074
Nov	1110	4280	1077.5	5390	
Dec	1090	4310		5370	

Question 8

	Qtr1	Qtr2	Qtr3	Qtr4
Year 1	57	85	97	73
Year 2	64	96	107	89
Year 3	76	102	115	95

Year	Quarter	3 MT	3MA
1	57		
	85		79.7
	97	239	85
	73	255	78
2	64	234	77.7
	96	233	89
	107	267	97.3
	89	292	90.7
3	76	272	89
	102	267	97.7
	115	293	104
	95	312	

Question 9

A shop keeps records of its monthly expenditure for advertising and its total monthly sales, I the last nine months, the records were as in the table below:

Advertising cost (000')'	41	42	34	36	45	38	39	52	35
Sales (000")	72	74	58	66	77	68	39	92	63

i) Find the least squares regression equation for the data.

ii) Use the regression equation above to predict the sales if the company plans to spi Frw 520,000 for advertising in the following month.

i) y = 2.534 + 1.7022x
Frw 885, 146.534 (thousands) or Frw 885,146,534.

Month	Sales	3MT	3MA
Jan	190		

Feb	250		286.7
Mar	420	860	410
Apr	560	1230	373.3
May	140	1120	300
Jun	200	900	253.3
Jul	420	760	430
Aug	670	1290	410
Sep	140	1230	373.3
Oct	310	1120	296.7
Nov	440	890	490
Dec	720	1470	

23.5. Forecasting

Types and characteristics of forecasts

Month	Sales	Month	Sales
Jan	190	Jul	420
Feb	250	Aug	670
Mar	420	Sept	140
Apr	560	Oct	310
May	140	Nov	440
Jun	200	Dec 720	

Below are the sales of a soda depot, in thousands of crates:

Range of forecast	Average time span	Application	■ Characteristic	Forecast method
Long term	5 or more years	Business planning: Product planning Research Plant location	Broad, general	Technological. Marketing studies. Economic analysis.
Intermediate (medium) term	Up to 2 years	Aggregate planning Capital, cash budgets Production and Sales planning	Estimate reliability and numerical data needed.	Time series. Regression. Correlation.
Short term	Less than 1 year. (1 day up to 1 year.)	Short run control: adjustment of production, purchasing, overtime decisions	Mainly at item level	Trend extrapolation. Product forecast. Smoothing techniques.

Quantitative forecasting methods

Forecasting methods can be developed through subjective (qualitative) quantitative (statistical approach). In this workbook, we shall only consider the quantitative forecasting methods only.

Quantitative forecasting methods use mathematical expressions or models to show the relationship between two variables. Time series models use time as independent variable to estimate the future dependent variable.

The first step in developing a quantitative forecasting model is to collect sufficient data on the past levels. Data far back in time is considered to be indicative of the conditions expected in the future. The effects of unusual or irregular events that is not expected to reoccur at regular intervals of time should be removed from the data. Such types of data distorts the pattern of the trend from generated data and may give false impression It is helpful to make a graphical display of the data to see if a pattern is present that ma, help to predict future values.

The time series models often give adequate forecasting tools if the variables in the series show a fairly consistent pattern over time and the conditions under which the pattern occurs are expected to continue. A time series is therefore a sequence of data co acted at equal intervals of time and arranged in the order of their occurrence daily, weekly monthly, quarterly or yearly. Sometimes a pattern may not be apparent in raw data, such data can be decomposed into components that reflect a pattern which are helpful in projecting data. There are four commonly recognised components **in** time series; its trend, seasonal cyclic and random. Of the four components in time series, only trend and seasonally components are easier to identify. Random -variations by definition are unpredictable so not easy to identify. Cyclic require many years of data to determine the degree of their repetitiveness.

Measures of forecast accuracy

The model that is appropriate for forecasting a pattern depends on the pattern to be projected and the forecast objectives.

When the random component of a time series has fluctuations that deviate substantially from the average level, it is better to smooth out the data by averaging several observations to get a basic pattern apparent. Simple moving average, weighted moving average and exponential smoothing are appropriate to produce final forecasts.

Time series decomposition is more appropriate if seasonal variations are evident in the data pattern and the effect of seasonality is to be included in the forecast. Time series decomposition can be used when the general trend in the pattern is horizontal or not. The concepts of both additive and multiplicative seasonal are more applicable in time series decomposition.

For example, since demand is influenced by many factors whose future values are not known, it would be unrealistic to expect a demand forecast values to be exact at any time Hence forecast estimation which implies forecast values may contain an errors.

A forecast error is the difference between forecasted value and the actual. A calculation of the average error made by forecast model over time provides how well the forecast matches the pattern of past data. This measure is often used as an estimate of how well the model will fit the pattern of the data it is trying to predict. Such measure provides a basis for comparison to see which model seem to better to use in forecasting. Four measures a commonly used:

i) Mean absolute deviation (MAD)
ii) Mean square error (MSE)
iii) Mean forecast error (MFE)
iv) Mean percentage error (MAPE)

Mean absolute **error**

Mean absolute error (MAD) is the mean errors made by fore cast model over a series of time periods without regard whether an error made was over estimate or under estimate. Sometimes MAD is called mean absolute error (MAE).To calculate MAD forecast value from actual value for each time period, change all the sign of Values to positive and sum up and divide the result by number of values used to get the sum.

By formula MAD = ⁺-Δ- where Yt = actual values in time t.

Ft = forecast value at time t. n = number of times being used. Mean square error

This is commonly used when many small errors are above or below the actual. They are located in such way that they average out one another, giving the impression of the best forecast

A method of measuring errors that gives large errors a large weight and a smaller weight is preferred. The mean square error provides this type of measure of forecast error. Multiplying each forecast error itself (squaring the errors) gives a large weight to a large weight to large errors and a small weight to small errors.

The MSE is found by squaring each of the series error made by forecast model, summing the squared error and dividing by the number of errors.

Mean forecast error (MFE)

A good forecast model should not only have small errors but it should be unbiased. An unbiased model makes positive errors negative. It does not have tendency to overestimate more than to underestimate. For unbiased models, positive errors almost cancel out with negative error. If the sums do not cancel out it means the forecast is biased The MFE is calculated by summing the forecast errors over a series period and dividing by this sum by number of errors used to compute the sum.

Mean absolute percentage error (MAPE)

Sometimes it is more informative to get relative percentage error. The relative error that a forecasting model makes can be measured by the mean absolute percentage using the formula.
To calculate MAD substitute forecast value from actual value for each time period, change all the sign of values to positive and sum up and divide the result by number of values used to get the sum.

By formula MAD = $\frac{\sum[At-Ft]\sum[At-Ft]}{n \qquad n}$ where At = actual values in time t.
Ft = forecast value at time t.
n = number of times being used. Mean square error

This is commonly used when many small errors are above or below the actual. They are located in such way that they average out one another, giving the impression of the best forecast

A method of measuring errors that gives large errors a large weight and a smaller weight is preferred. The mean square error provides this type of measure of forecast error. Multiplying each forecast error itself (squaring the errors) gives a large weight to a large weight to large errors and a small weight to small errors.

The MSE is found by squaring each of the series error made by forecast model, summing the squared error and dividing by the number of errors.

By formula MSE = $\frac{\sum[At-Ft]2\sum[At-Ft]2}{n \qquad n}$

Mean forecast error (MFE)

A good forecast model should not only have small errors but it should be unbiased. An unbiased model makes positive errors negative. It does not have tendency to overestimate more than to underestimate. For unbiased models, positive errors almost cancel out with negative error. If the sums do not cancel out it means the forecast is biased The MFE is calculated by summing the forecast errors over a series period and dividing by this sum by number of errors used to compute the sum

MFE=$\frac{\sum(At-Ft)\sum(At-Ft)}{n \qquad n}$

Mean absolute percentage error (MAPE)

Sometimes it is more informative to get relative percentage error. The relative error that a forecasting model makes can be measured by the mean absolute percentage using the formula.

MAPE= $\left(\frac{100}{n}\right)\sum\left|\frac{At-Ft}{At}\right|\left(\frac{100}{n}\right)\sum\left|\frac{At-Ft}{At}\right|$

Example 1
The table below shows the demand and forecast.of bread at hot spot bakery in one week**.**

	Monday	Tuesday Wednesday		Thursday	Friday	Saturday
Demand	120	130	110	140	110	130
Forecast	125	125	125	1251	125	125

Calculate:

i) MAD;
ii) MSE;
iii) MFE; and
iv) MAPE

(i) Demand	(ii) Forecast	(iii) Deviation A - F	(iv) Absolute deviation \ A - F \	(v) Squared error { A – FY)2	(vi) Percentage error $(\frac{A-FA-F}{A\quad A})100$	(vii) Absolute Percentage error. $(\frac{A-FA-F}{A\quad A})100$
120	125	-5	5	25	-4.17	4.17
130	125	5	5	25	3.85	3.85
110	125	-15	15	225	-13.64	13.64
140	125	15	15	225	10.71	10.71
110	125	-15	15	225	-13.64	13.64
130	125	5	5	25	3.85	3.85
		-10	60	750		4986

i. **MAD =** $\frac{60}{6} = 6\frac{60}{6} = 6$

ii. **MSE=** $\frac{750}{6} = 125\frac{750}{6} = 125$

iii. **MFE:** $-\ \frac{10}{6} = -1.67-\ \frac{10}{6} = -1.67$

iv. **MAPE=** $\frac{49,86}{6} = 8.31\%\frac{49,86}{6} = 8.31\%$

Time series decomposition

Times series decomposition is the separation of the overall series into basic components that are more likely to have recognisable and more predictable patterns. These components can be projected in the future and be combined to form a forecast:

Basically, there are four components: trend, cyclic, seasonal and random irregular). Decompositions assumes that these components act independently. It is also assumed that what caused them to can in the past will continue to operate in the same way even in the future.

The time series decomposition can be used to separate or decompose a time series into trend, seasonal and irregular components. Decomposition method can be use for forecasting and also to get a better understanding of the time series. Many business and government agencies use time series decomposition to create de-seasonalised time series.

Time series models fall into one of the categories depending whether their components are expressed as sums or products.

The adoptive decomposition model, assumes the values of **Y** equals the sum of the four components at any period t.

$Yt = Trendt + Seasonalt + Irregulan\,values$ $Yt = Trendt + Seasonalt + Irregulan\,values$ for period t. where Trend t = trend value at time period t. Seasonal t = seasonal value at time period t, Irregulan = irregular value at time period t.

By assuming that the components are additive we are in effect taking components to be independent of each other.

The multiplicative decomposition model takes the form, in which the relationship between components are expressed in the form $Yt = Trendt\ x\ Seasonalt\ x\ Irregulan$,$Yt = Trendt\ x\ Seasonalt\ x\ Irregulan$, where values are defined in the same way as in additive model. This form assumes components are related to each other and yet still allows for components to result from different basic causes.

The main purpose of de-seasonalising data is to remove the seasonal component from the data in order to get a clear picture of nonseasonal trend component. This process is referred to as de-seasonalising time series.

Sometimes the additive model of time series is more appropriate than multiplicative model. The isolation of the components for additive model proceeds in a similar way to multiplicative model except the differences are used instead of ratios. A moving average for the series is first determined to smooth out the seasonal and irregular components. The seasonal and irregular components are isolated by taking the difference between each observed value and corresponding **Yt** and corresponding moving average. The values of **St** + It are the seasonal deviations, which are then averaged to remove irregular effects. These averages are then adjusted so that they sum up to zero. The adjusted averages are called seasonal deviations and represent seasonal component

De-seasonalised or seasonally adjusted values are obtained by subtracting corresponding seasonal deviations from time series observation for Yt, yielding the trend - cyclic irregular component $T_t + C_t$ + It

In case of sales, in a given time period, $\mathbf{Y_t}$ will comprise both a trend element T_t and a seasonal variation S, which will depend on the time. Furthermore, there ma. be random or Irregular h. which affect sales in a particular time period but which are unlikely to be repeated in future, e.g. dispute, change in tax policy new competitors in the neighbourhood etc.

These are considered residual elements which are unpredictable but would affect sales but had nothing to do with seasonal variations. The calculations appropriate for the time series analysis use the following steps: Incorporating seasonality in forecast is useful when both trend and seasonality exist "in" time series data. » Trend time series help to obtain trend estimates for a desired period using trend equation

Deviations of actual from trend values and seasonality help to get trend estimates by adding or multiplying (assuming additive or multiplicative model) Average deviations in the time period help to obtain seasonally adjusted values used to de-seasonalise data.

Calculation of a trend line

By definition, the trend is the underlying long-term movement of a variable with seasonal variation that is the repetitive in nature. Trend values are usually shown together with actual values. The most common trend method is that of centred moving average. The trend is generally smooth and shows long term positive growth or future growth.

Method and steps in using the additive model to calculate seasonally adjusted values.

Step 1: Graph the time series in order to get the visual impression of the trend of growth whether positive or negative:

If the trend is positive the graph is upward sloping; and If the trend is negative the graph is downward slopping.

Step 2: Calculate the moving averages.

Step 3: Subtract trend values in each period from actual figure to produce period deviations.

Step 4: Produce a table of deviations from which you calculate mean deviations from each period (quarter).

Step 5: Total the average deviations and check whether they do or do not add up to zero. If they do not, you will need to adjust them. Total calculated d vided by period and subtract this from each of the deviations. This gives seasonally adjusted variables in each quarter.

Step 6: Compare your graph and your statistics to check whether they match

At this stage, you may plot trend values on the same graph with original values

Example 6

The table below shows quarterty consumption of water in thousands of Rwandan Francs for the past three years in a coffee industry.

Quarterly amount in thousands of Rwandan Francs

Year	1st	2nd	3rd	4 quarters
2010	74	76	74	80
2011	82	68	50	62
2012	70	74	70	82

Required

i) Calculate the trend values using 4 centred moving averages.

ii) Plot the original data and trend values on the same graph.

iii) Using additive model adjust seasonal indices and smooth consumption figures.

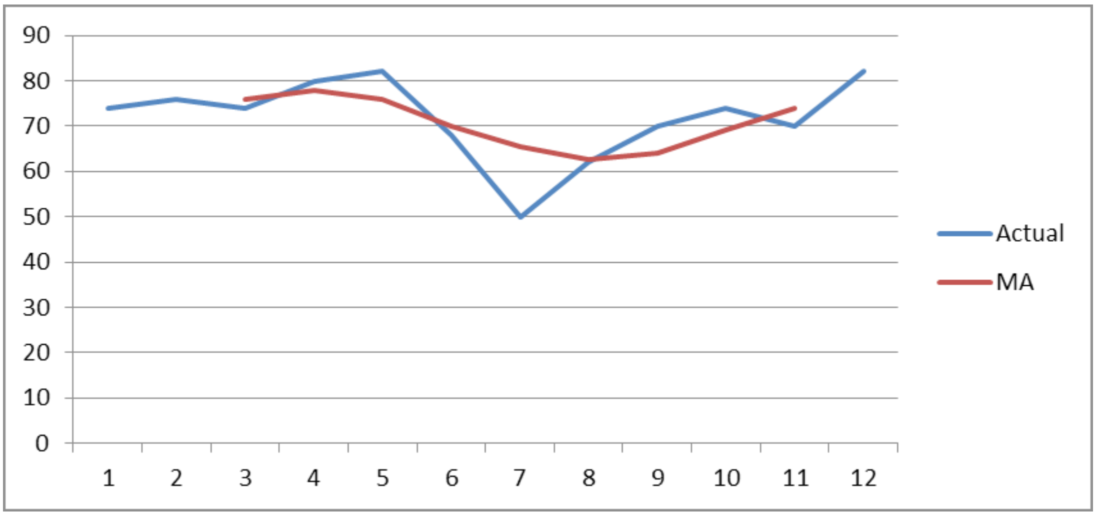
Solution

Year	Quarter	consumption	Quarterly	Moving	Centered	Deviations
year		(FRW '000)	moving total	average	average	actual - trend
2010	*1*	*74*	-			
	2	*76*	-			
	3	*74*	*304*	*76*		
	4	*80*	*312*	*78*	*77*	*- 3*
2011	*1*	*82*	*304*	*76*	*77*	*3*

	2	68	280	70	73	9
	3	50	262	65.5	67.75	0.25
	4	62	250	62.5	64	- 2
2012	1	70	256	64	63.25	-1.25
	2	74	276	69	66.5	3.50
	3	70	296	74	71.5	2 5
	4	82	-			

Computation of de-seasonalised data

Year	Quarter	Consumption..	Seasonal	De-
	*		indices	seasonalised
			‘	data
2010	1	74	6.25	67.750
	2	76	1.375	74.625
	3	74	-8.500	82.500
	4	80	0.875	79,125
2011	1	82	6.25	75.750
	2	68	1.375	66.625
	3	50	-8.500	58.500
	4	62	0.875	61.125
2012	1	70	6.250	63.750
	2	74	1.375	72.625
	3	70	8.500	78.500
	4	82	0.875	81.125



Forecasting a time series

Time series is not used for analysis only but it is also used for forecasting future values of a variable under analysis. To forecast a series, it requires two sets of forecasts.

Forecast variable = forecast of trend +forecast of seasonal variation.

That is to forecast the future value for the variable we need to forecast the future value of the trend and future value of seasonal variation.

Linear regressions assume that:

- the trend will continue uncharged;
- seasonal factors will also continue unchanged; and
- there will be residual elements in any period.

Regression analysis is used primarily for the purpose of prediction. The aim is tc use a statistical method to predict using a dependent based on the value of one independent variable.

The least squares method is the most appropriate in this case

Months	Production in 1000's kg
January	80
February	90
March	92
April	83
May	94
June	99
July	92

The table below shows amount ®f sugarcane produced by out growers in (1000's kg) j Lugazi-area-in 7 months in 2012:

a) Use least squares method to find the trend equation.

b) Show the production and trend equation on the same graph.

Months	X	Production (y)	Month -April x	X	xy	Yc	Y-Yc
Jan	1	80	-3	9	-240	84	-4
February	2	90	-2	4	-180	86	4
March	3	92	-1	1	-92	88	4
April	4	83	0	0	0	90	-7
May	5	94	1	1	94	92	2
June	6	99	2	4	198	94	5
July	7	92	3	9	276	96	-4
N=7		630	0	28	56		0

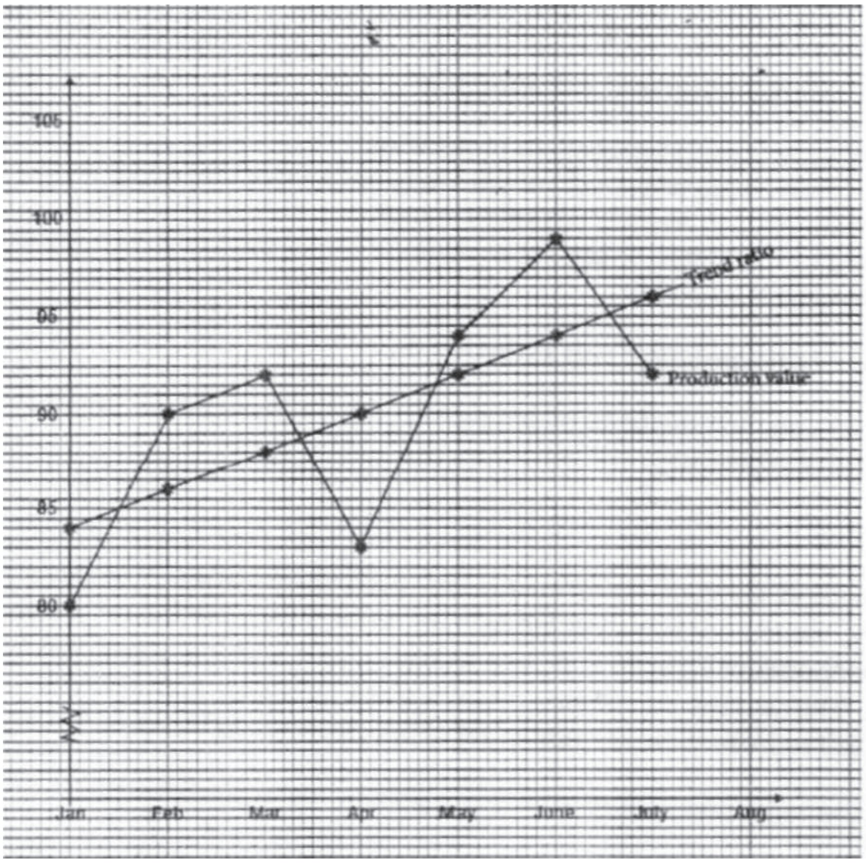
c) Use the trend equation to predict the production for August.

Where Ex=0 $a=\frac{\sum Y \sum Y}{N\ N}$, b= $a=\frac{\sum Y \sum Y}{X2\,X2}$

Solution:

a) Least squares trend equation: y = 82 + 2x.

Using the equation Yc= a+bx



c) Predicted value for August is 98.

Interpolation and extrapolation

Interpolation and extrapolation are methods used to determine missing values in recorded data. Interpolation refers to missing data values in between two neigh bough points th=: falls within tabulated values. Extrapolation is used when a data values falls outside the tabulated data.

Generally, there are three assumptions used in computing values by interpolation a-: extrapolation.

There is no sudden jump or fall or break changes in the data in the series period to another. This implies that data must be free of all sorts of random irregular fluctuations.
The rate of change of the data values remains constant. This means that data CS" be presented by a function of a certain degree. It is essential that a fairly good number of recordings should be available so that effects of past behaviour can be clearly visible on interpolated value-.Therefore, interpolation and extrapolation*assumes a functional linear relationship of the independent and the dependent data values. The methods used for interpolation and extrapolation, are all based on a functional linear relationship. For any function in terms of x it means that we can always find a definite value of y corresponding to x, that is,

$y = f(x)$.

a) Uses of interpolation and extrapolation

Helps in making forecasts or projections.

Helps in cases where part of data may have been destroyed orr lost which may create gaps in already tabulated values.

Helps to estimate or predict a value for some past or future period for which data is not available

Method of interpolation and extrapolation

The most common methods of computing interpolation and extrapolation are either graphical or algebraic.

Graphical

This method involves representing the given data values by means of a graph. Usually, values are paired in the form (x, y) and then plotted on a squared paper. The independent variable is plotted along x - axis and the dependent variable on y - axis. The various coordinates obtained are joined by a smooth free hand curve which will represent the general trend of the relationship between the two variables. Using the regression equation, it is possible to make predictions. The process of fitting a straight line to bivariate data is known as linear regression. The main aim of linear regression is to model the relationship between the independent and dependent variables by using the equation of a straight line. Once we have the equation, we can use it to make predictions.

Example 8

The data below shows the performance and time spent on revision of a sample of 10 students in one of the Training Institution in Kigali

Time hours (x)	4	36	23	19	1	11	18	13	18	8
Marks % (y)	41	87	67	62	23	53	61	43	65	52

Required:

i) Determine the regression equation relating hours of revision and marks scored.

ii) Use the regression in to predict the marks scored by a student who revised for 0. 8, 30, and 50 hours.

iii) Comment on each result in (ii).

x	y	xy	X^2
4	41	164	16
36	87	3132	1296
23	67	1541	529
19	62	1178	361
1	23	23	1
11	52	572	121
18	61	1098	324
13	43	559	169
18	65	1170	334
8	52	416	64
=151	=553	=9853	=3215

By applying lest aquare formula we obtain

Hence, the regression line for the trend is y = 31 + 1.6x.

ii) Using the regression equation y = 31 + 1.6x.

0 hours of revising will lead a student to obtain y = 31 + 1.6 x 0 = 31%. 8 hours of revising would lead a student to score y = 31 + 1.6 x 8 = 44c -30 hours of revising would lead a student to score y = 31 + 1.6 x 30 = 79%. 50 hours of revising would lead a student to score y = 31 + 1.6 x 50 = 11%.

iii) Comment: The 8 and 30 hours of revision and corresponding scores f within the range of marks, and are a result of interpolation. Here it is safe to use interpolation while 50 hours of revision and corresponding score of 1'1 % fall outside the range of marks and is a result of extrapolation.

Note: Interpolation is a suitable method of approximation within the range of data to be interpolated and extrapolation is a method of approximation of data /values outside the range of values.

CHAP 24.

NETWORK ANALYSIS

24.1. Study objectives

By the end of this chapter, you should be able to:

- explain the terms network, activity, event, dummy activity, critical path, float, cost slopes, dangler and lead time;
- draw a network and Gant chart for a given project;
- demonstrate the concept of control and planning technique called network analysis;
- use dummy activities in drawing networks correctly; and
- determine the critical path and explain PERT analysis.

24.2. Network analysis

Network analysis is the organised application of systematic, logical planning, scheduling and controlling practical situations, where many separate tasks can take place simultaneously or consecutively, such that it is difficult to establish the relationship between the separate jobs. The technique can be applied to any purposeful chain of events involving the use of time, labour and physical resources. It is usually related to large industrial or commercial projects of a complex nature where the scale of operation gives rise to correspondingly greater financial and administrative problems. Network analysis involves the following three basic principles:

- It defines the job to be done.
- It integrates the various activities in a logical time sequence.
- It controls the progress of the project plan.

24.3. Definition of terms

Network

A network is a graphic representation of a project's operation and is composed of activities and events (nodes). In network events are identified by numbers i.e.1,2,3,4.... while activities are represented by letters A, B, C, D ....

Activity

Any individual operation that consumes time, money or other resources and has an end and beginning is called an activity. An activity is represented in a network by an arrow, the tail of which represents the beginning and head the completion of the activity.

In a project to build a house, build a wall, dig foundations, etc. represent activities

Event

This is a point in time and indicates the start or finish of an activity or activities, for example, foundation made, wall built, etc. represent events. An event is represented in a network by a circle or node, like this:

Dummy activity

This is an activity which does not consume time or resources. It is used merely to show clear, logical dependences between activities, so as not to violate the rules for drawing networks. It is represented on a network by a **dotted arrow.**

Note that dummy activities are not usually listed with the real activities but become necessary as the network is drawn.

Critical path

An unbroken chain of activity arrows in a network which connects the initial event to some other is called a path. The path consisting of the sequences of those events and activities that require the maximum time in the completion of the project is known as **critical path** of the project. It is the chain of activities which has the **longest duration.** The activities associated with the critical path are called critical activities.

Float

Float or slack means extra time over and above its duration which a non – critical activity can consume without delaying the project. There are three types of float:

Total float

This is the amount of time a path of activities could be delayed without affecting the overall project duration. It is given by the total time which is available for performance of the activity minus the duration of the activity.

Total float = latest head time - earliest tail time - activity duration

Free float

This is that part of the total float which does not affect the subsequent activities in other words free float is the time by which the completion of an activity can be delayed beyond the earliest finish time without affecting any subsequent activity.

Free float= earliest head time - earliest tail time - activity duration

Independent Float

This is the amount of time an activity can be delayed when all preceding activities are completed as late as possible and all succeeding activities completed as early as possible. Independent float therefore does not affect the float of either preceding or subsequent activities.

Independent float = earliest head time - latest tail time - activity duration

Cost slopes

This is the average cost of shortening an activity by one-time unit (day, week, month as appropriate). The cost slope is generally assumed to be linear and is calculated as follows:

Cost slope = $\frac{\text{crash cost} - \text{normal cost crash cost} - \text{normal cost}}{\text{normal time} - \text{crash timenormal time} - \text{crash time}}$

Or cost slope: is defined as the ratio of difference between crash cost and normal cost and difference between normal time and crash time.

Crash time

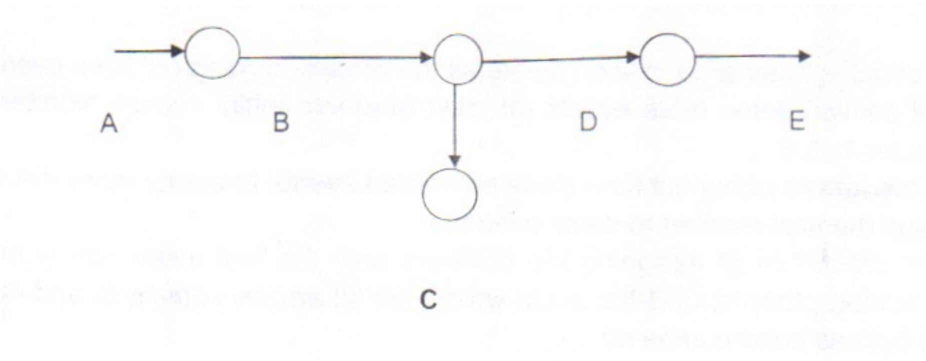
Crash time is time taken by the activity when additional resources, overtime and other special measures are taken to speed up (crash) the activity.

Lead time

It's the time between starting and finishing. The time interval between the start of an activity or process and its completion for example the time between ordering goods and their receipt or between starting manufacturing of a product and its completion.

Dangler

It is that activity other than the initial (starting) and final (ending) that does not have any successor events. To avoid such situations, it should be kept in mind that all events except the first and last of the whole project must have at least one entering and one leaving activity. See fig below: normal time - crash time



The activity C does not have any successor event. Such a situation is known as dangling in network analysis. Or danglers are activities which do not link into the overall project.

Drawing of networks

For the construction of a network, we make use of the following rules:

Each activity is represented by one and only one arrow. This implies that no single activity can be represented twice in the network.

Not two activities can be identified by the same end events. This implies that there must be no loops in the network.

Time flows from left to right. Arrows pointing in opposite direction must be avoided.

Arrows should be kept straight and not curved or bent.

Avoid arrows which cross each other.

Use dummies freely in rough draft but final network should not have any redundant dummies.

Every node must have at least one activity preceding it and at least one activity following it except for the very beginning and at the very end of the network. The

beginning node has no activities before it and the ending node has no activities following it.

Only one activity may connect any two nodes. This rule is necessary so that an activity can be specified by giving the numbers of its beginning and ending nodes.

After constructing the network, next is to assign a number to every event and to place it inside the node circle. The number sequence should be such that it reflects the flow of the network. The following steps are followed in numbering:

The starting event has outgoing arrow(s) and no incoming arrow is numbered '1'or 'O'

If the initial event is numbered 0' then delete all the arrows coming out from event 'O'. This, will convert some more events (at least one) into initial events. Number these events as 1,2, 3...

Remove all the arrows going out from these numbered events to create more initial events. Assign the next number to these events.

Continue- the procedure of assigning the numbers until the last event has been assigned a number, that is, until the event which has all arrows coming in and no arrow going out has been numbered.

The above procedure of assigning numbers is known as **Fulkerson Rule.**

Example 25.1

The following are the various elements of a project to erect a steel works for a shed.

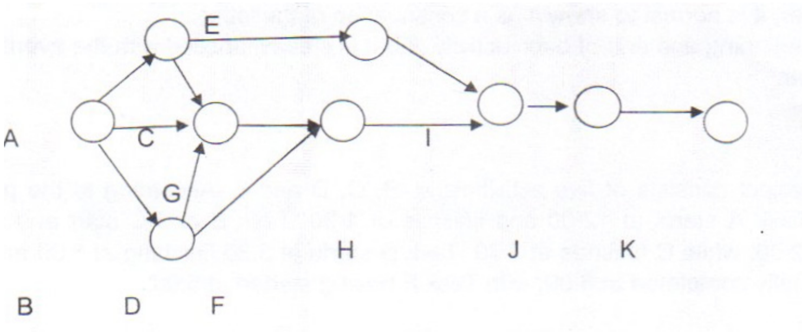
Activity code	Description	Predecessor
	Erect site workshop -	-
	Fence site	-
	Bend reinforcement	A
	Dig foundation	B
	Fabricate steel work	A,C
	Install concrete plant	B
	Place reinforcement	C,D
	Concrete foundation	G,F
	Paint steel work	E
	Erect steel work	H,I
	Give finishing touch	J

Required:

Draw a net for the above project of erection of steel works shed.

Solution:

The resulting network is shown below:



Note:

Activities may be identified in several ways.

Typical of the methods to be found include:

Shortened description of the job, for example, plaster wall, order umber etc.
Alphabetic or numeric code, for example, A, B, C, etc or 60, 61, 69, etc.
Identification by the tail and head event numbers, for example, 1 -2,2 — 3, 3-5 etc.

Gant Charts

Gant charts (or bar charts) are useful in project planning and control. Allocating resources to the project is called loading the network and is represented on a diagram termed as **Gant chart.** This chart helps to analyse and balance the availability and requirements of various resources. On a Gant chart, the activities are represented by lines having lengths proportional to the duration of each activity.

Each activity on the Gant chart is identified by its linking events. It has been assumed that each activity starts at its earliest time. The dotted line shows the total float of each activity.

Main features of a Gant Chart

i. The horizontal scale represents time. From this scale the duration of each activity and the total project time can be seen quite clearly.

ii. The critical activities are shown as one straight line at the top of the chart.

iii. The other activities are shown below. It is assumed in each case that they are stated as clearly as possible. The dotted lines represent the total float of each activity. If one non-critical activity is dependent solely on the completion of another, it is normal to show it as a continuation of the latter.

iv. The beginning and end of each activity / float is clearly labelled with the event number.

Example 25.2

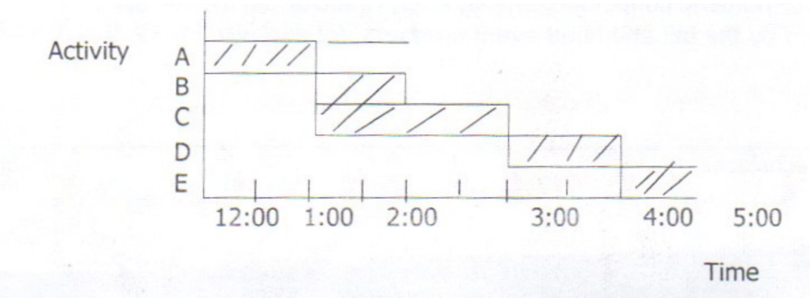
A certain project consists of five activities; A, B, C, D and E. According to the project manager, Task A starts at 12:00 and finishes at 1:30. Task B and C start at 1:30, B finishes at 2:30, while C finishes at 3:30. Task D starts at 3:30 finishing at 5:00 and the project is finally completed at 6:00, with Task E having started at 5:00.

Required:

Construct a Gant chart for the five activities.

Solution:

Gant chart for the activities.



5. Advantages of network analysis
   i. Network analysis helps to determine the objective of the project.
   ii. The method enforces planning, because data from many sources must be collected and collated before being logically put together to give the network.
   iii. Areas of responsibility are specifically defined, the relationship between activities is clearly shown and the network reveals the interactions of all participants.
   iv. The technique provides for simple communication and, therefore, easy to apply because the network diagrams and charts are easily understood by non-specialists.
   v. Control is simplified, because network analysis permits the use of management by exception, where by the management need act only when the situation is out of control.
   vi. The technique is equally applicable to large and small scale operations.
   vii. The system lends itself easily to computers and many computer manufacture rsprovide standard packages of network analysis routines with their equipment.

24.4. Limitations of network

These depend on the method, there are two methods:

Critical Path Method (CPM)

i. CPM is based on the assumption of known time for each of the activity in the project which may not be true in real life.
ii. For determining the time estimates, CPM does not incorporate statistical analysis.
iii. When certain changes are introduced in the network, the entire evaluation of the project is to be repeated and a new critical path is found.

Project Evaluation and Review Technique (PERT)

i. A PERT is a chart is a graphic representation of a project's schedule, showing the sequence of tasks, which tasks can be completed simultaneously and the critical path of tasks that must be completed on time in order for the project to meet its completion deadline. The chart can be constructed with a variety of attributes, such as earliest and latest start times for each task, earliest and latest finish times for each task and slack times between tasks.

PERT is based on the time estimates rather than known time for each activity.

ii. It emphasises only time and not costs.

iii. It is not practicable for routine planning of recurring activities.

iv. In PERT, the calculations based on probabilities are carried out on the assumption of independent activities. The distribution of total time is assumed to be normal but in real life this may not be true.

v. For active control of a project, it requires frequent updating and revising of the PERT calculations.

Determination of critical paths

Network analysis is a general term given to certain specific techniques which can be used for planning, management and control in projects. Critical path, project path and review are two techniques considered in this chapter. The major use of networks is to help in scheduling, determining how long the project will take and when each activity should bed one. The expected project duration (mean) is determined by finding the longest path through the network. The longest path from start to finish is called critical path.

Example 24.3

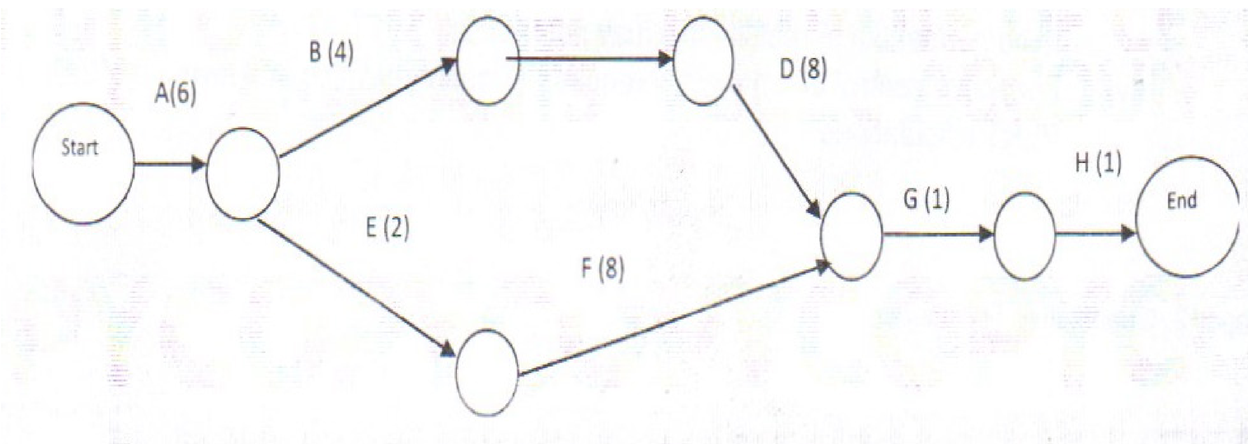
Rice Development Agency is in the process of computerising its systems. The activities and duration times required for the exercise are listed in the table with precedent activities.

	Activity	Preceding activity	Duration (hrs)
A	Develop systems specification	-	6
3	Buy and deliver Hardware	A	4
C	Assemble and test Hardware	B	6
D	Install Hardware	C	8
E	Specify software	A	2
F	Purchase and deliver software	E	8
G	Test the system	D, F	**1**
H	Manager tests the software	G	**1**

Required:

a) Draw a network for all activities.

b) Determine the critical path and state critical activities.



.The first activity A(6)

After it has been completed B(4) and E(2) can begin but it will take BCD 4+6 + 8 = 18hrs to handle hardware activity and 2 + 8 = 10 hrs to do software E and F.

Because activity A takes 6 hours and the hardware BCD will be ready for G in 6 + 18 =24hrs, software in 6 +10 = 16hrs.

However, both hardware and software activities must be finished before G. The earliest for G will be 24 hrs. Therefore, the project expected duration time will be 24 + 2 = 26 hrs.

Notice there are two paths:

A-B-C-D-G-H = 26hrs

A-E-F-G-H = 18hrs

A B C D G H are critical activities which make the critical path. The critical path will take 26hrs. There are two standard ways to draw a network - critical path (CP) and project (program) evaluation and Review technique (PERT)

i) Activity on arc (arrow) AOA which is used in CP.

ii) Activity on node (rectangle) AON which is used in PERT.

Example 24. 4

The following table shows activities and times to improve an overgrown home garden.

	Activity	Time (5 min)	% Activity
A	Clear garden	5	-
B	Measure area	1	-
C	Design patio	2	B
D	Choose fencing	1	B
E	Buy pots and plants	3	A,C
F	Plant all pots	1	E
G	Purchase slabs	1	C
H	Construct garden	6	C,D,G

Required:

a) Draw:

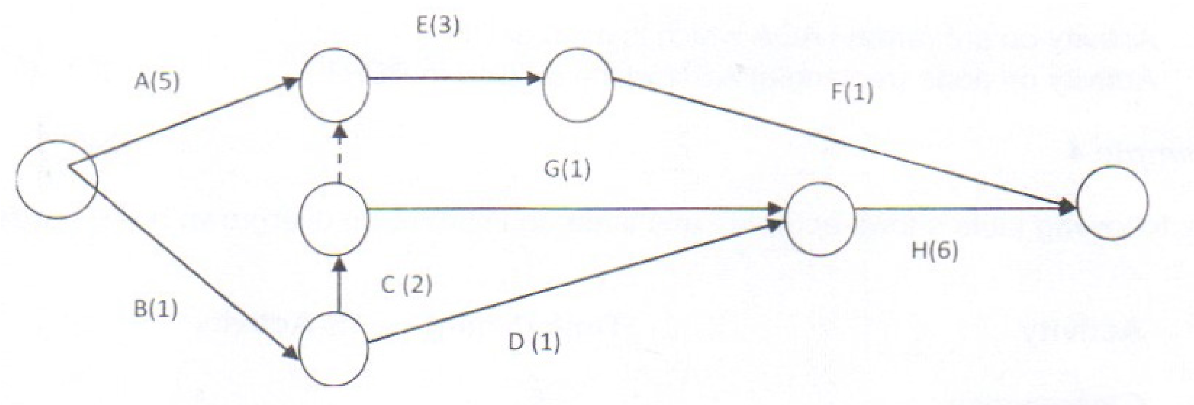
i) an activity network on arrow and determine the critical path; and

ii) an activity network on node and determine the critical path.

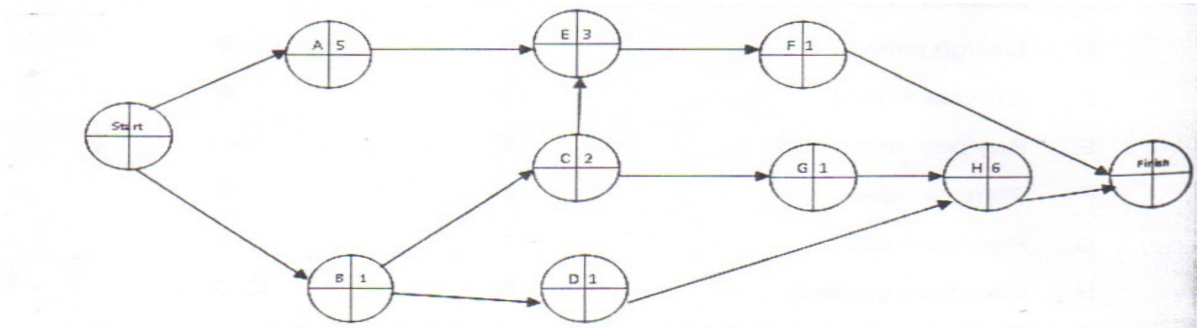
b) Calculate event times, forward and backward, and, hence, determine critical path.

Solution:

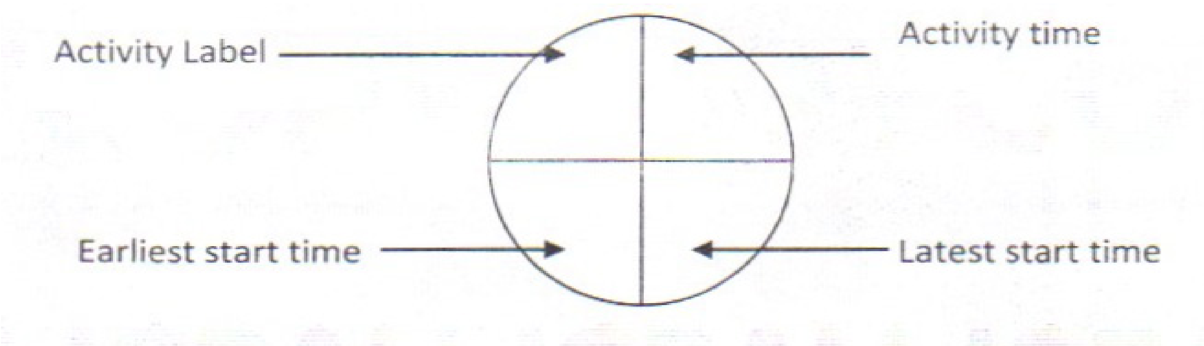
a) i) Activity on Arrow (AOA)

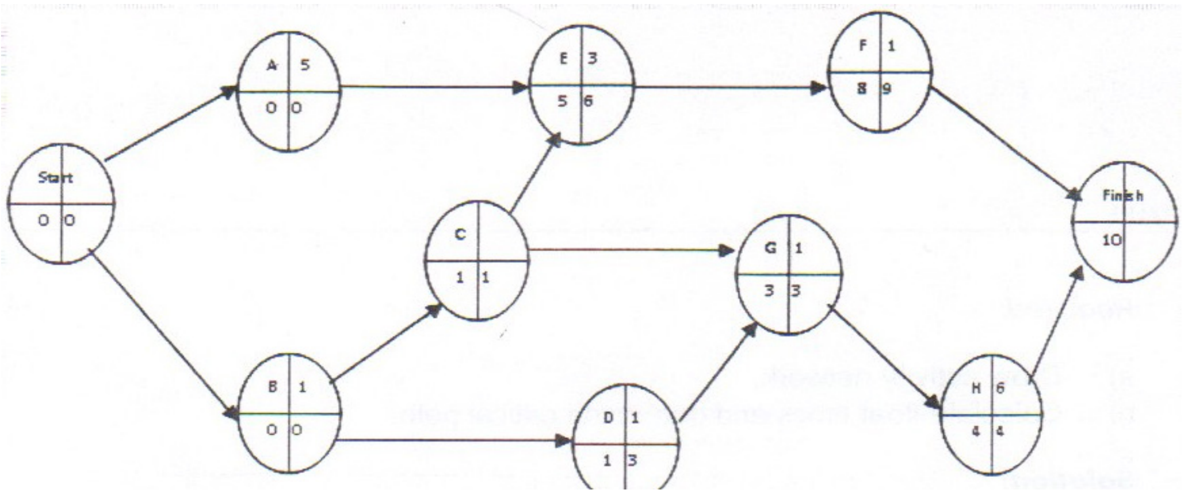


ii) Activity on Node (AON)



b) Using activity on Node





Forward pass is from start to finish. This is a method to add the early start and finish to each in a network diagram.
Backward pass is from finish to start. This is a method to add the late start and finish to each in a network diagram. This makes the network diagram look complicated, but it gives a lot of valuable information.

The critical path is that sequence of activities where there is no slack, the longest

Path BCGH = 10hrs.

Example 24.5

The following table shows activities duration time and preceding activity in a project:

Activity	Preceding activity	Duration (days)
A	-	3
B	-	5
C	-	4
D	A	2
E	B	3
F	C	9
G	D, E	8
H	B	7
I	H, F	9

Required:

a) Draw activity network.

b) Calculate float times and determine critical path.

Solution:

Table showing float of each activity:

Activity A	L.S 9	E. S 0	Total float LS-ES 9	Note
B	1	0	1	
C	0	0	0	CP
D	12	3	9	
E	11	5	6	
F	4	4	0	CP
G	14	8	6	
H	6	5	1	
1	13	13	0	CP

Critical activity s are C - F - I

Critical path = 4 + 9 + 9 = 22days

Note: Both critical path method (CPM) and project (program) evaluation and review technique (PERT) are called critical path methods because both use critical path to compute expected duration of the project, using early start times, late start times and slack.

But PERT was developed for application where there is uncertainty associated with the nature and duration of activities. Therefore, PERT is based on the assumption that an activity's duration follows a probability distribution instead of being a single value.

Three time estimates are required to compute the parameters of an activity's duration time.

Pessimistic time (t_p) - the time the activity would take if things did not go well.

Most likely time t_m- the normal time by consensus (best estimate of activity duration).

Optimistic t_0- is the time the activity would take if things go well.

The three time estimates are related in the form of a Beta probability distribution.

Using this distribution, the mean or expected time (t_e) and variance (v_t) of each activity are computed with three time estimates using the following formulas.

Mean (expected time) te = $\frac{tp+4tm+tctp+4tm+tc}{6 \qquad 6}$

Variance $Vt = \sigma^2 = (\frac{tp-to}{6})Vt = \sigma^2 = (\frac{tp-to}{6})$

Furthermore, in PERT analysis, the following are important:

Draw the network and analyse the paths through the network and find critical path.
The length of the critical path is the mean of the project duration probability distribution which is assumed normal.
The standard deviation of the project duration probability distribution is computed by adding the variances of the critical activities and taking the square root of the sum.
The probability computations can now be made using the normal distribution table $Z = \frac{x-\mu}{s} Z = \frac{x-\mu}{s}$ where, μ= tp = project mean time,

σ= project standard deviation

x = (projected) specific time

Self-test questions

Question 1

a) Define the terms:

i. Gant chart.

ii. Network.

b) A project consists of five activities A, B, C, D, and E that satisfy the following precedence relationships:

i. Neither A nor B has any immediate predecessors.

ii. A is an immediate predecessor of C.

iii. B is an immediate predecessor of D.

iv) C and D are immediate predecessors of E.

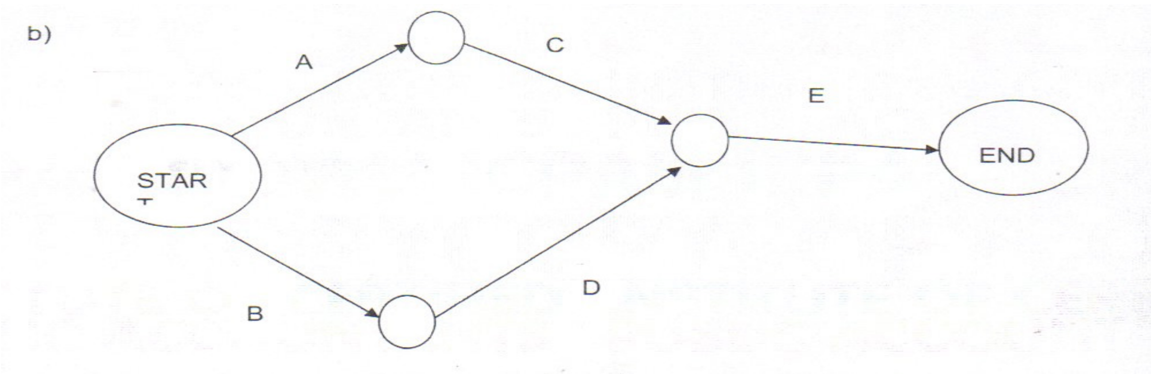
Required:

Draw a network for this project.

Solution:

a) i) The definition of a Gant chart can be found in Chapter 27, section 4.

ii) The definition of network can be found in Chapter 27, section 2a).



Question 2

Define the terms:

a) Float.

b) Cost analysis.

Solution:

a) The term float is defined in Chapter 27, section 2f).

b) The definition of cost analysis can be found in Chapter 27, section 2g).

Question 3

Describe the following:

i) Dummy activity.

ii) Latest starting time.

iii) Lead time.

iv) Dangler.

v)

Solution:

i) Dummy activity is defined in Chapter 27, section 2d).

ii) Latest starting time is the result obtained by subtracting activity duration from late finishing time.

iii) Lead time is defined in Chapter 27, section 2i).

iv) Dangler is defined in Chapter 27. section 2j).

Question 4

Write short notes on the following terms as used in network analysis:

i) Predecessor task.

ii) Successor task.

iii) Earliest finish time (EFT).

iv) Latest start time (LST).

Solution:

i) Predecessor task is a task which takes place before any other activity.

ii) Successor task is a task that takes place after other tasks.

iii) Earliest finish time (EFT) is described in Chapter 27, section 2f).

iv) Latest start time (LST) is described in Chapter 27, section 2f).

Question 5

Compare and contrast the project evaluation and review technique (PERT) with critical path method (CPM).

Solution:

PERT and CPM are specialised methods of project management techniques and scheduling tools that allow project managers to plan, manage and control tasks in projects. They are jointly referred to as network analysis models or critical path analysis techniques. Although they are historically different, they are usually in conjunction with each other.

PERT and CPM are used in various organisation projects to plan, organise and monitor project management related activities. PERT charts are management tools that help in decision making. They are diagrams that represent the flow of activities through a process, highlight all dependent tasks and events, display the sequence of activities from the start to the end of the project and show the critical path of the project. Activities are represented by boxes and arrows. In CPM network diagrams activities show the sequence of activities in terms of cost and time.

PERT is more suitable for research and design, where a project is being done for first time and estimates of duration are uncertain, CPM is better suited for routine in projects where time and cost estimates are accurately calculated. Both PERT and CPM aim at avoiding wastage and surprises.

Question 6

Consider the following activities shown for constructing a garage shade:

Activity	Duration (days)
Prepare foundation	7
Make and position door frame	1
Lay drains floor base and screed	15
Install fittings	8
Erect walls	10
Plaster ceiling	2
Erect the roof	5
Install door and windows	8
Fit fitters	2
Paint walls	3

Required:

a) Prepare a precedence table for activities.

b) Draw activity network.

Question 8

a)

Activity	Duration (days)
Prepare foundation	-
Make and position door frame	-
Lay drains flow base and screed	-
Install fittings	E
Errect wall	A, B
Plaster ceiling	D, G
Errect the route	E
Install door and windows	G
Fit fitters	C. F
Paint walls	I

Question 7

The following data shows activities duration and precedence activities:

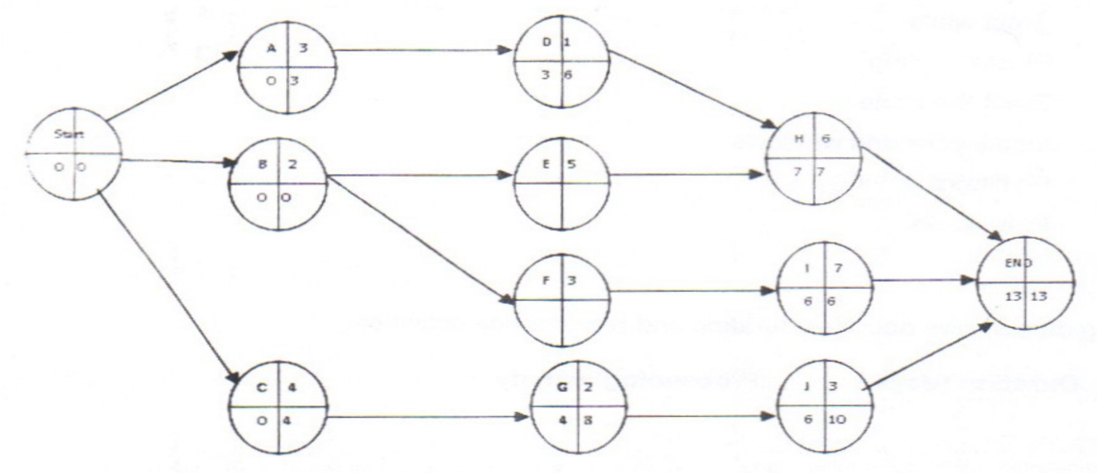
Activity	Duration (days)	Proceeding
		-
	2	-
	4	-
	1	A
	5	B
	3	B
	2	C
	6	DE
	7	F
	3	G

Required:

a) Draw an activity network on node.

b) Indicate the earliest and latest start times in the nodes.

Solution:



Question 8

The activities and time needed to put up a shelf in a room are indicated below:

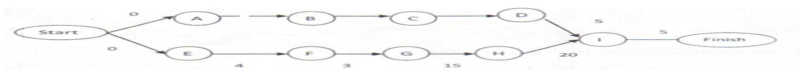
Activity	Time(min)	Proceeding activity
Mark positions of screw holes on the wall	10	-
Drill holes **in** the wall	9	A
Hammer in the wall plugs	2	B
Fix shelf brackets	5	C
Cut shelf to correct length	4	-
Drill holes in the shelf	3	E
Sand paper and vanish shelf	15	F
Allow vanish to dry	20	G

Fix itself to brackets	5	DH

Required:

Draw activity network.

Solution:



Question 9

The table shows the precedence activities in a project. Assuming project team member have five working days in one week and all tasks start on schedule.

Task	Description	Duration(days)	Predecessor(s)
A	Needs analysis	5	
B	Designing	15	A
C	Programming	25	B
D	Phone	15	B
E	Hardware fixing	30	B
F	Integrating into the system	10	C D
G	Testing	10	E. F
H	Training how to use	5	G
1	Market	5	H

Required:

Determine the:

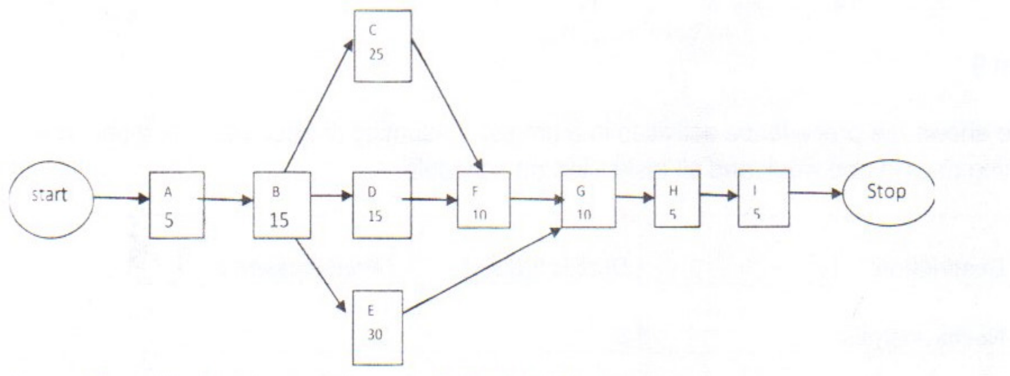
i. critical path of the project;

ii. mean duration of the project; and

iii. identify any non-critical tasks.

Solution:

The critical path can be ascertained from the network.



i) The critical path runs through A, B, C, F, G, H and I.

ii) Mean duration (5 + 15+25+10+10 + 5 + 5) =75 days.

iii) Non- critical activities B, D and E.

Question 10

a) i) Describe any two advantages of a Gantt chart,

ii) Describe one main application of a PERT chart.

b) A systems analysis and design project has been defined to contain the following sevenactivities.

Required:

	Activity Name	Time (weeks)	Immediate Predecessors
1	Collect requirements	2	
2	Analyse processes	3	1
3	Analyse data	3	2
4	Design processes	7	2
5	Design da! A	6	2
6	Design Product	1	3,4
7	Design reports	5	4,5

Solution:

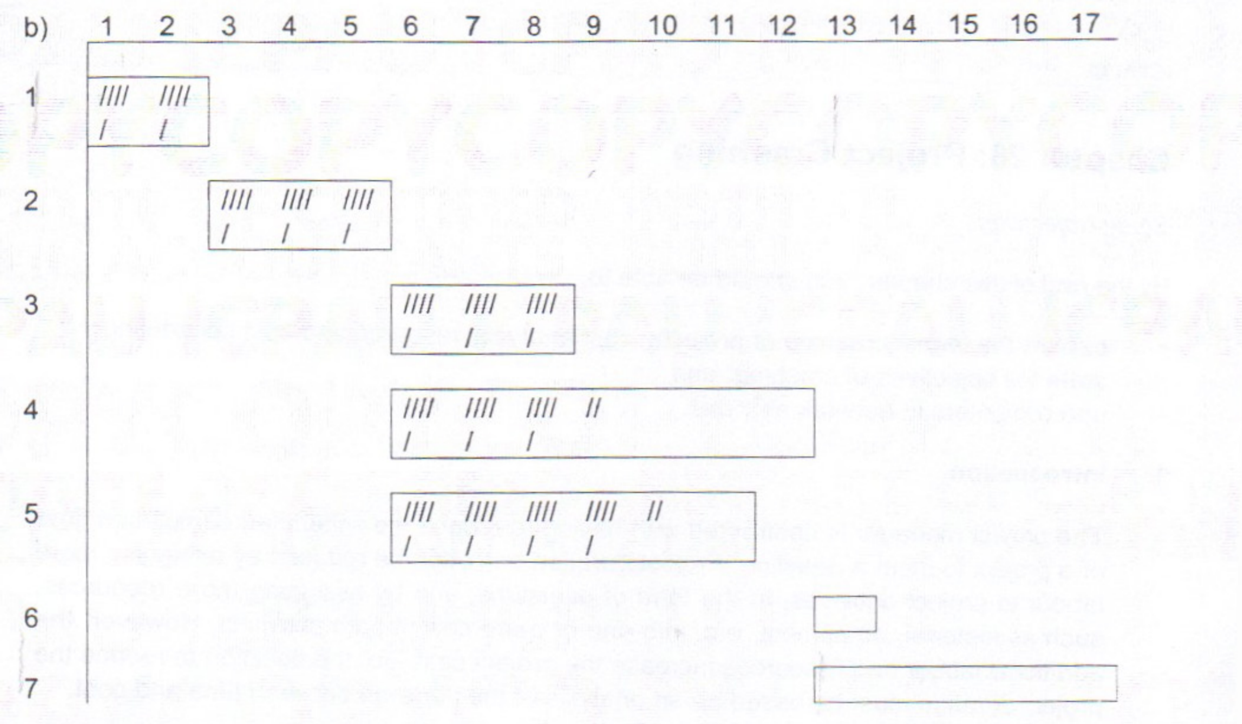
a) i) The advantages of Gantt charts

shows overlapping tasks and task duration,

provides simple graphical tracking of progress

iii) The main application of PERT charts is that they are useful for large projects withhigh task dependencies; shows task sequence & dependencies (not task duration); and emphasises critical path identification.

b)



///// - Represents shading to show how much of each task is complete.

24.5. Project Crashing

1. Introduction

The project manager is confronted with having to reduce the scheduled completion time of a project to meet a deadline. Project duration can often be reduced by assigning more labour to project activities, in the form of over time, and by assigning more resources, such as material, equipment, etc. into one or more critical path activities. However, the additional labour and resources increase the project cost. So, the decision to reduce the project duration must be based on an analysis of the trade-off between time and cost.

2. Project crashing

Project crashing is a method for shortening the project duration by reducing the time of one or more of the critical project activities to less than its normal activity time.

Objectives of crashing

To reduce the scheduled completion time, in order to reap the results of the project quickly.

As project continues over time, the team consumes indirect costs.

There may be direct financial penalties for not completing a project on time.

3. Cost analysis

In most projects, there are direct and indirect costs:

Direct costs include costs of labour, materials and equipment. These costs increase if the project time decreases.

Indirect costs include costs of rent, interest and utilities. These costs increase if the project time increases.

The goal of cost analysis is to determine the optimal time to perform the project that minimises the sum of indirect and direct costs.

4. Resource scheduling

It is the allocation process of scarce resources among competing activities

5. Methods of crashing

There are various methods of project schedule crashing, and the decision to crash should only take place after a carefully analysis of all the possible alternatives.

a) Increasing the resources

There are a number of standard and typical approaches to attempting to crash a project schedule. One of the most commonly utilised methods involves increasing the assignment of resources on schedule activities. This essentially means decreasing the time it takes to perform individual activities by increasing the number of people working on those activities. For example, if it takes Omony six hours to complete an activity, it would logically take both Omony and Kalisa three hours to complete the same activity.

However, adding resources is not always the best solution. Sometimes it ends up taking more time in the long run. For instance:

the new resources may not be familiar with the tasks at hand, so they will probably be less productive than the current;

who will guide the new members up the learning curve? Usually it will be the most productive members of the team who could themselves be working to get the task finished more quickly: and

being available does not equal being qualified. The best accountant available will not help if you need a computer programmer. Sometimes extra hands are only tangentially qualified for the work, and even if the new resources have the right skills, they may not be of the same calibre as the current team members.

With that in mind, maybe adding resources isn't the best method for crashing your project.

b) Fast tracking

Another solution may be fast tracking, which involves overlapping tasks which were initially scheduled sequentially. For example:

splitting long tasks into short and smaller units, thereby, squeezing more work into a shorter period of time;

reducing lag times between tasks: and

Reducing the scope to eliminate less important tasks.

Note:

Sometimes the best method is some combination of resource addition and schedule activity alteration. For instance, adding additional, qualified people to the taste be completed earlier and re-assign members with less experience to tasks that do not productive, one may re-schedule projects so that several items can be worked on at once instead of sequentially.

Because businesses are more complex today than ever before, project managers must become more rational in their decision making by using the most effective tools and techniques. Before deciding to crash a project, make sure you've looked at all of the possible options and thoroughly evaluated cost analysis models. That way you can achieve the greatest results for your efforts.

6. When to crash projects

The key to project crashing is attaining maximum reduction in schedule time with minimum cost. Quite simply, the time to stop crashing is when it no longer becomes cost effective. A simple guideline is to:

crash only activities that are critical crash from the least expensive to most expensive; and crash an activity only until it reaches its maximum time reduction; it causes another path to also become critical; and/or it becomes more expensive to crash than not to crash

Crashing is reducing project time by expending additional resources.

Crash time is the amount of time an activity is reduced.

Crash cost is the cost of reducing the time of an activity.

7. Project crashing

The goal of crashing is to reduce project duration at minimum cost. To reduce project duration white minimizing the cost of crashing, the project team should estimate the required time, required cost, crash time, and the crash cost for each of the activities. And then the team can estimate total crash time, total crash cost, the crash cost per week to reduce project duration at minimum cost.

8. Assumptions

The time required to complete each activity is known and fixed.

Activity times can be reduced at additional cost.

The time specified for completing an activity is the normal time.

The costs, of completing the activity in each of these times are known.

Example

Two accountants, P and Q, at their consultancy firm plan to complete their project workin 25 weeks. Once completed, they anticipate that they will receive an average of Frw 150,000 per week for the first three years that the product is available. They carefully' consider each activity, the possibility of reducing the activity time and the associated costs. They estimate that the normal costs are Frw 150,000 per week for activities that they themselves as consultants do, and either more or less for activities that they contract out. They obtain the following estimates

	Time (weeks)		Cost (Frw)	
Activity	**Normal**	**Crash**	**Normal**	**Crash**

A	3	1	100,000	300,000
B	4	3	400,000	600,000
C	2	2	200,000	200,000
D	6	4	300,000	600,000
E	5	4	250,000	380,000
F	3	2	150,000	300,000
G	7	4	450,000	810,000
H	5	4	300,000	360,000
I	8	5	800,000	128,000

Required:

a) Find the total cost of performing the project in the normal time of 25 weeks.

b) Calculate the cost per week.

c) Find the critical path(s).

d) Find the saving that the consultancy realises by reducing the project completion time.

Solution:

Activity	Time(weeks)		Saved	Cost (Frw)			
	Normal	Crash		Normal	Crash	Increase	Per week
A	3	1	2	100,000	300,000	200.000	100.O30
B	4	3	1	400,000	600,000	200.000	200. COO
C	2	2	0	200.000	200.000	-	-
D	6	4	2	300.000	600.000	30C.0OO	150,000
E	5	4	1	250.00C	380,000	130.000	130,000
F	3	2	1	150,000	300.000	150.000	150,000
G	7	4	3	450.00C	810,000	360.000	‘20.000
H	5	4	1	300,000	360,000	60.000	60.000
I	8	5	3	800,000	1,280,000	480.000	160.000
				2,950-OOC	4,830.000	1.820,000	

a) The total cost of performing the project in the normal time of 25 weeks is obtained by summing the normal costs for each of the activities.

= 100,000 + 400,000 + 200,000 + 300,000 + 250,000 + 150.000 + 450,000 + 300,000 + 800,000

= Frw 2,950,000

b) Cost per week $= \frac{\text{crash cost} - \text{normal cost}}{\text{normal time} - \text{crash time}}$ b) Cost per week $= \frac{\text{crash cost} - \text{normal cost}}{\text{normal time} - \text{crash time}} = \frac{\text{Cost increase}}{\text{time saved}} \frac{\text{Cost increase}}{\text{time saved}}$

c) If the normal times are replaced with the crashed times, then using the criterion of crashing from the least expensive to most expensive, it is easy to show that there are two critical paths based on crashed times:

i) A-C-E-G-I

ii) A-C-E-H-I

d) The saving that the consultancy realises by reducing the project completion time from 25 to 16 weeks is 9 x 150,000 = Frw 1.350,000.

But the additional cost of reduction is Frwn 4,830,000 - Frw2,950,000 =Frw1,880,000. Hence, it is not economical to reduce all of the activities to their crashed times. Crashing all activities means that both critical and non critical activities are crashed (expedited). However, there is clearly no economy inexpediting noncritical activities. Hence, it is likely that there is a solution with a project completion time of 16 weeks that costs less than Frw4,830,000.

10. How to identify the activities to crash

If the cost increase is less than Frw150,000, the reduction is clearly economical and additional reductions should be considered. But if the cost of the reduction exceedsFrw150,000, further reductions are not economical. The key point is that in order to reduce the time required to complete the project by one week, it is necessary to reduce the time of an activity along the current critical path, or activities along the current critical paths if more than one path is critical. Reducing the time for noncritical activity will not reduce the project time.

In the previous example, we have the following:

Project time	**Critical path(s)**	**Critical activities**	**Current time**	**Crashed time**	**Cost to reduce by one week**
25	A-**C**-**E**-G-**I**	A	3	1	100,000
		C	2	2	**-**
		E	5	4	130,000
		G	7	4	120,000
		I	8	5	160,000

The least expensive activity to reduce is A. We can reduce activity A to **1** week without introducing any critical paths. Because the cost of each weekly reduction is less thanFrw**150,000,** it is economical to reduce A to its minimum time, which is one week. At this point we have the following:

Project time	**Critical path(s)**	**Critical activities**	**Current time**	**Crashed time**	**Cost to reduce by one week**
23	**A-C-E-G-I**	**A**	**1**	**1**	**-**
		C	**2**	**2**	**-**
		E	**5**	**4**	**130,000**
		G	**7**	**4**	**120,000**
		I	**8**	**5**	**160,000**

The next activity to reduce is G. The critical path will remain the same until G is reduced to five weeks. (Consider reducing G by one week at a time to be certain that no additional paths become critical). When G is reduced to five weeks, both paths A-C-E-G-I and A-C-E-H-I are critical. Reducing G from seven weeks to five weeks, results in the following:

Project time	Critical path(s)	Critical activities	Current time	Crashed time	Cost to reduce by one week
21	A-**C**-E-G-I	A	**1**	**1**	-
	A-**C**-E-H-I	**C**	**2**	**2**	-
		E	**5**	**4**	**130,000**
		G	**5**	**4**	**120,000**
		H	**5**	**4**	**60,000**
		I	**8**	**5**	**160,000**

In order to further reduce the project time to 20weeks, it is necessary to be certain that we make the reduction along both critical paths. Although H has the least marginal cost, we should not reduce it. This is because if we reduced H to four weeks, only the critical path A-C-E-H-I is reduced and not the path A-C-E-G-I. If we reduce both H and G, the increase in the direct cost is Frw180,000**,** which is not economical.

10. How to reduce project time further

The activities A, C, E and I lie simultaneously along the both critical paths. Hence, a reduction in the activity time of any one of these four activities will result in a reduction of the project time. Among these activities, E can be reduced from five to four weeks for under Frw 150,000. Making this reduction, we obtain:

Project time	Critical path(s)	Critical activities	Current time	Crashed time	Cost to reduce by one week
20	**A-C-E-G-I**				
	A-C-E-H-I	**A**	1	1	**-**
		C	2	2	
		E	4	4	**-**
		G	5	4	120,000
		H	5	4	60,000
		I	8	5	160,000

At this point the cost of reducing the project by an additional week exceeds Frw 150,000.So, the optimal solution is reached. The reduction from 25 weeks to 20 weeks costs a total of Frw (100,000 *2) + (120,000x2) +130,000 = Frw 570,000 in additional direct costs, and results in a return (saving) of Frw (150,000x5) = Frw 750,000. If all cost and time estimates are correct, the consultancy firm would have realised a saving of Frw 750,000 - 570,000 = Frw 180,000.

11. Cost analysis

In most projects, there are direct and indirect costs.

Direct costs include costs of labour, materials and equipment. These costs increase if the project time decreases.

Indirect costs include costs **of** rent, interest and utilities. These costs increase **if** the project time increases.

The goal of cost analysis is to determine the optimal time to perform the project that minimises the sum of indirect and direct costs.

12. Resource scheduling

It is the allocation of scarce resources among competing activities. Program evaluation and review technique/ critical path method **(PERT/CPM)** assumes that each activity has available all the resources (money, personnel, equipment, etc.) needed to perform the activity in the normal way (or on a crashed basis). However, many projects have only limited resources for which the activities must compete. A major challenge in planning the project then is to determine how the resources should be allocated to the activities, so as to minimise its duration.

1. Cost scheduling

Projects activities have their earliest and latest start times. A project manager takes into account these times in costing. He needs to know how much money is required to cover each week's expenses, as well as the cumulative amount, assuming the project can stick to the earliest start time schedule. Then, to indicate how much flexibility is available for delaying expenses, he uses PERT/cost to do the same thing when the individual activities begin at their latest start times instead.

2. Use of computers in network analysis

Program evaluation and review technique/ critical path method (PERT/CPM) continues to evolve to meet new needs. At its inception in the late 1950s, it was largely executed manually. The project network sometimes was spread out over the walls of the project manager. Recording changes in the plan became a major task. Communicating changes to crew supervisors and subcontractors was cumbersome. The computer has change all of that.

For many years now, PERT/CPM has become highly computerised. There has been a remarkable growth in the number and power of software packages for PERT/CPM that run on personal computers or workstations. *Project management software* (for example. Microsoft Project) is a standard tool for project managers although there are dozens of software packages for project management.) This has enabled applications to numerous projects each involving many millions of dollars and thousands of activities. Possible revisions in the project plan now can be investigated almost instantaneously. Actual changes and the resulting updates in the schedule, etc..are recorded virtually effortlessly. Communications to all parties involved through computer networks and telecommunication systems also have become quick and easy

Self-test questions

Question 1

a) Distinguish between resource scheduling and cost scheduling.

b) What are the advantages of using a computer in project evaluation and review technique/ critical path method (PERT/CPM)?

c) Describe the methods used in crashing of projects.

Solution:

a) The distinction can be found in Chapter 28, sections 3 and 4.
b) The solution to this can **be** found in Chapter 28, section 14.
c) The methods can be found in Chapter 28, section 5.

Question 2

A project has the normal and the crashed costs and times as given in the table below:

Activity	Immediate predecessor	Normal time (weeks)	Crashed time (weeks)	Normal cost ($)	Crashed cost ($)
A	-	3	2	200	250
3	A	5	3	600	850
C	A	1	1	100	100
D	B, **C**	4	2	650	900
E	B	3	2	450	500
F	E, D	3	2	500	620
G	F	2	1	500	600
H	E,D	4	2	600	900

Required:

a) Find the direct cost of the project after reduction by considering successive reductions in the project time of one week.

b) Given that the indirect costs are $150 per week, find the optimal:

i) project completion time; and

ii) total project cost.

Solution:

a) $3,600

b) i) 14 weeks

ii) $5,320

Question 3

A project has the normal and the crashed costs and times as given in the table below:

Activity	Normal time (days)	Crashed time (days)	Normal cost ($)	Crashed cost ($)
A	32	26	200	500
B	21	20	300	375
C	30	30	200	200
D	45	40	500	800
E	26	20	700	1,360
F	28	24	1.000	1,160
G	20	18	400	550

Required:

If the indirect costs amount to $100 per day, determine the optimal:

a) time to complete the project; and

b) project completion cost.

Solution:

a) 178 days

b) $22.745

Question 4

A project to be implemented by a local NGO in Karamoja consists of activities A, B. C, D, Eand F, inclusive, with the following sequential relationship: A must follow C and E, B must follow

A and F. D must follow B, F must follow C. Activities C and E may take place at the same time and similarly activities A and F. The project manager has come up with the following implementation Schedule:

Activity	Time (weeks)		Crash cost (Frw '000)
	Normal	Crash	
A	4	1	240
B	5	3	200
C	4	2	120
D	1	1	-
E	2	2	-
F	3	1	80

The benefactor of the project requires the project to be completed in 13 weeks. A penalty of Frw 200,000 per week for excess time is imposed while a bonus of Frw 90,000 per week is granted for earlier completion.

Required:

a) Draw a logical network based on normal durations for the project.

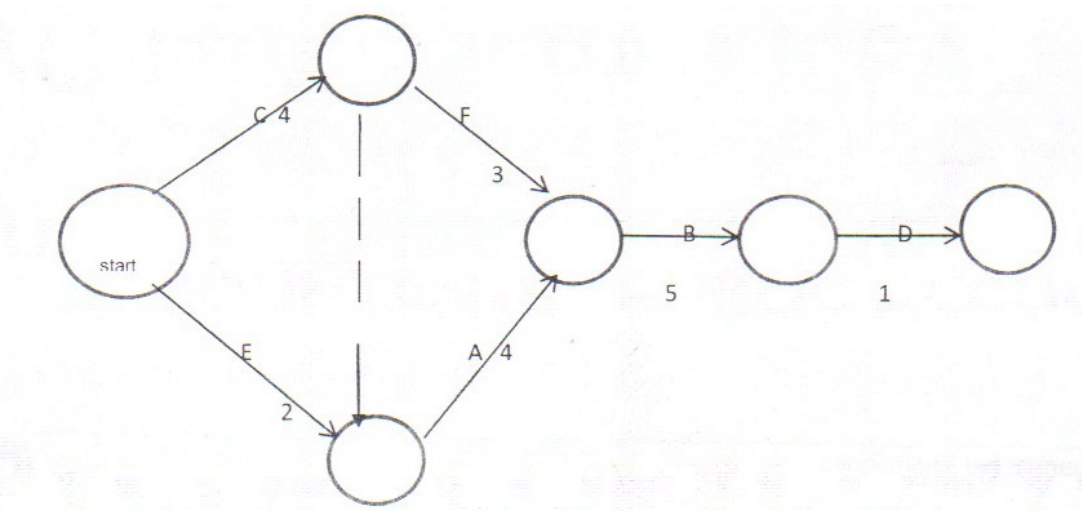
b) Determine the critical activities and the normal duration of the project.

c) Find the data which would enable one to plot the cost/time curve for the project.

d) Explain whether the normal duration should be changed and the effect it would cause.

Solution:

a)



b) Critical activities are: C, A, B and D. Normal duration: 4+4+5+1=14 weeks.

c) Cost/time data:

Duration (weeks)	Additional cost (Frw)	Bonus (Frw)	Net cost (Frw)
14	200,000	0	200,000
13	60,000	0	60,000
12	120,000	-90,000	30,000
11	200,000	-180,000	20,000
10	300,000	-270,000	30,000
9	400,000	-360,000	40,000
8	520,000	-450,000	70,000

d) Duration of the project could be shortened to 11 weeks with additional cost of $20,000 by substituting the crash duration for C for the normal and then crashing by 1 week on activity A. Any other attempt to reduce further attracts additional expense and hence uneconomical.

Question 5

a) Explain the following terms as applied to network analysis

i) Normal time and crash time.

ii) Crash cost and cost slope.

b)The below shows details of a project being undertaken by Jobas Ltd.

	Preceding activity	Duration (Weeks)		Cost (Frw million)	
		Normal	Crash	Normal	Crash
A	-	3	2	300	380
B	-	5	2	250	460
C	-	4	3	180	240
D	A	4	2	240	340
E	A	8	5	315	450
F	D, B	6	4	220	340
G	C	7	5	280	404

(Adapted from 2015 Public Accountants Examinations Board)

i) Draw an activity network diagram for the project, giving the order of the activities and clearly showing the EST and LST for the activities.

ii) Determine the critical path and state the normal time for the project

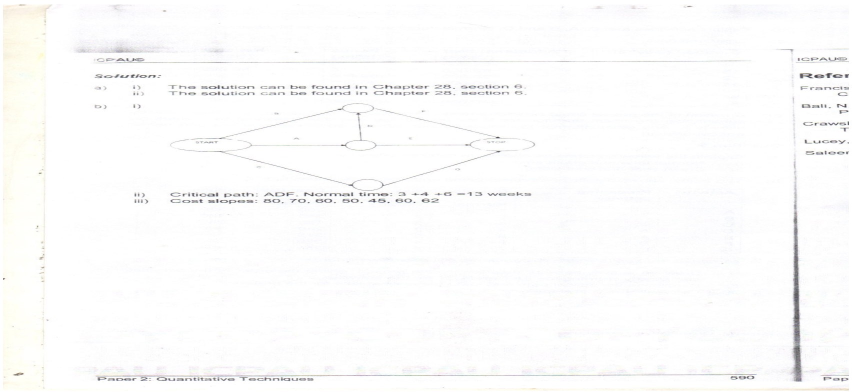
iii) Calculate the cost slopes

Solution:

a) i) The solution can be found in Chapter 28, section 6.

ii) The solution can be found in Chapter 28, section 6.

b) i)



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Solution:

a) i) The solution can be found in Chapter 28, section 6.
ii) The solution can be found in Chapter 28, section 6.

b) i)

ii) Critical path: ADF. Normal time: 3 +4 +6 =13 weeks
iii) Cost slopes: 80, 70, 60, 50, 45, 60, 62

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ii) Critical path: **ADF.** Normal time: 3 +4 +6 =13 weeks

iii) Cost slopes: 80, 70, 60, 50, 45, 60, 62

CHAP 25.

DECISION THEORY

25.1. Study objectives

By the end of this chapter, you should be able to:

define terms: state of nature, event, decision alternative, pay off;

explain decision making types and decision rules;

arrange information in a payoff table or decision tree;

find the expected pay off and decision alternative; compute and interpret expected monetary value results;

give advantaged and disadvantage of decision trees; and determine redundancy in decisions.

25.2. Decision making

Decision making is an integral part of human life. We are faced with situations every day where we have to make decisions, choosing among many alternatives available to us under different situations. These decisions may be simple such as how to spend a daily routine or major decision such as changing a job, or introducing a new product in the market. As long as a choice has to be made out of two or more alternatives it is considered a decision.

Statistical decision theory is concerned with determining which decision from a set of possible alternatives, is optimal for a particular set of conditions. Decision analysis can be used to develop an optimal decision strategy when a decision maker is faced with several decision alternatives and uncertain or risk filled pattern of future events.

25.3. Elements of decision making

There are three main components to any decision

a) Decision variables

These are the choices or alternatives available to the decision maker. They are controllable variables which are within the domain of the decision makers and can be changed or manipulate. Different values can be assigned to a decision variable to give a decision maker different courses of action to choose from. Example in case of making a decision about an investment, the decision variable would be different areas in which to invest, the amount to invest and the timing of such investment

b) Uncontrollable variables

These are states of nature which are not under the control of a decision maker. Uncontrollable variables are all factors whose situation are beyond control of the decision maker example may be price of competitor's products, costs of raw materials, political nature, inflation, etc.

c) Payoff

Payoff is an output variable that results from the efforts of the decision maker. Payoff is needed to compare each combination of decisions alternatives and state of nature.

25.4. Analysis of decision making

The structure of decision making can be considered under: objectives, strategies and uncertainties.

a) Objectives

A decision maker must clarify what he/she intends to achieve, in financial terms.

This may be to maximise profits or revenue, or market share, or simply minimize costs of production, exploration or introducing a new product.

b) Strategies

Having formulated objectives, the course of action to be followed to achieve the objectives must be identified. Some of the strategies may include investment decisions, that is, decisions to introduce a new product, new payment to workers.

c) Uncertainties

These are conditions which might prevail called states of nature. These are circumstances mainly which the decision maker has no control. In order to assess the effectiveness of different strategies in meeting objectives under uncertainty a measure in monetary terms must be used.

Solving decision models consists of finding a strategy for a course of action or expected relative value. The criterion for any strategy is always to maximise decision maker's expected value.

25.5. Types of decision making

There are three types of situations in which decisions can be made:

i. decisions under certainty:

ii. decisions under risk; and

iii. decisions under uncertainty.

a) Decisions under certainty

This is also known as decisions with perfect information. The conditions of certainty are known and payoffs can be determined accurately. An example of decision making with certainty is like to buy a new car. Once a decision to buy a new car is made, there are courses of alternatives in which payment should be made by cash, hire purchase or loan.

It is possible to calculate the total cost for each alternative course of action. In this example, you choose the one which gives the lowest cost.

b) Decision making under risk

A situation of risk occurs when there is incomplete reliable information. In this case, three area number of possible outcomes for each possible state of nature, where the probability of each outcome is known or is assigned. Then an expected value for each strategy is calculated. The strategy that yields the best (highest) expectedvalueis selected. Usually the decision problem is **presented in a table or matrix form.**

A decision problem is characterised by decision alternatives, state of nature and the resulting pay off.

Decision alternatives are the different possible strategies the decision maker can employ.

State of nature refers to future events which ultimately affect decision results.

States of nature should be defined so that they are mutually exclusive and contain all possible future events that could affect the result of potential decision.

Decision theory problems are generally represented by a payoff table or a decision tree.
A pay off table shows results from a specific combination of decision alternatives and state of nature. Pay offs can be expressed in terms of profit, cost, time, distance, etc.
A decision tree is a diagrammatic chorological representation of the decision problem.

Each decision tree has 2 types of nodes:

Circular nodes that correspond to state of nature, and rectangular nodes which correspond to decision alternatives.

The branches leading to each circular node represent different state of nature while branches leaving each rectangle represent different decision alternatives

The ends of a limb of a tree are pay offs attained from a series of branches making up that limb.

Example 1

A developer is planning to construct a structure in the urban centre. He/she must decide the type of structure to set up: large, medium, or small complex.

State of nature. The state of nature could be defined as low demand or high demand of the structure in the market.

Alternative. The alternatives could be deciding to build small, medium or large structure.

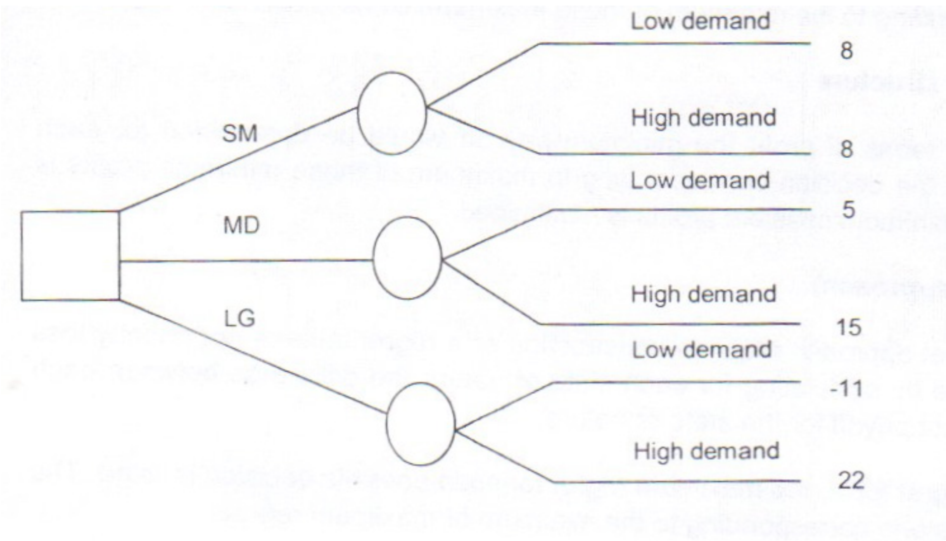
Pay offs. The profit for each alternative, under each alternative, under each potential state of nature.

Payoff table

Alternatives	**State of nature**	
	Low	**High**
Small	8	8
Medium	5	15
Large	-11	20

Figures in million Rwandan Francs

Decision tree



Where SM represents small complex

MD represents Medium complex

LG represents large complex

Decision making without probabilities

There are three commonly used criteria for decision making:

Optimist approach (maximax).

Conservative approach (pessimist).

Minimax regret approach (opportunist).

The maximax looks at the best that could happen under each action and then choose the best pay off, if the payoff is in terms of cost, the decision with lowest cost would be chosen. If the payoff is in terms of profit the decision with highest profit would be chosen. The maximum person looks at the worst that could happen under each action and chooses the action with the largest payoff of the worst. For each decision, the worst payoff is listed and then the decision corresponding to the best of the worst payoff is selected. Hence, the worst possible payoff is maximised.

If the payoff was in terms of cost then the maximum cost would be determined for each decision, corresponding to the minimum of those maximum costs is selected.

Decision making structure

If the payoff is in terms of profit the minimum pay off would be determined for each decision and then the decision corresponding to maximum of those minimum profits is selected, hence, minimum possible profits is maximised.

Minimax (regret approach)

The minimax regret approach requires construction of a regret table/or opportunity loss table. This is done by calculating for each state of nature the difference between each pay off and the best payoff for the state of nature.

Then using the regret table, the maximum regret for each possible decision is listed. The decision chosen is one corresponding to the minimum of maximum regrets.

From the above example, determine maximax pay off best.

Alternative	**State of nature**		**Profit**
	Low	**High**	
Small	8	8	8
Medium	5	15	15
Large	-11	22	22

Maximax pay off = 22

Conservative

Alternative	**State of nature**		**Lowest Profit**
	Low	**High**	
Small	8	8	8
Medium	5	15	15
Large	-11	22	-11

Maximum pay off = 8

If minimax regret is used, determine best payoff for a state of nature and create a regrettable.

From the pay of table:

Alternatives	**State of nature**	
	Low	**High**
Small	8	8
Medium	5	15
Large	-11	22

Best payoff for low = 8 and Best payoff for high = 22.

Decision making with probabilities

Expected value approach.

If probability information regarding the state of nature is available, one may use Expected Monetary Value (EMV) approach. In this approach expected payoff for each decision is calculated by summing up the products of payoff under each state of nature occurring. The decision yielding the best expected return is chosen. The expected monetary value of a decision alternative is the sum of the weighted pay offs for the decision alternatives. Expected monetary value (EMV) of a decision alternative is given by the formula.

EMV (di) = $\sum_{j=1}^{N} = 1Ps_1v_{ij}\sum_{j=1}^{N} = 1Ps_1v_{ij}$

Where N = number of possible states of nature

P(si) = probability of state of nature

Vij = the payoff corresponding to the decision alternative (di) and state of nature s.

Example 2

The Prince of Hope Restaurant is planning to open a new restaurant on Kigali Ma 'Street. He has three different models, each with different seating capacity. Prince of Hop estimates the average number of customers per hour will be 80, 100 or 120 in the three models, respectively. The payoff for table for the three models will be as follows:

Average number of customers per hour

Model	**S_1 =80**	**S_2 = 100**	**S_3=120**
A	10,000	15,000	14,000
B	8,000	18,000	12,000
C	6,000	16,000	21,000

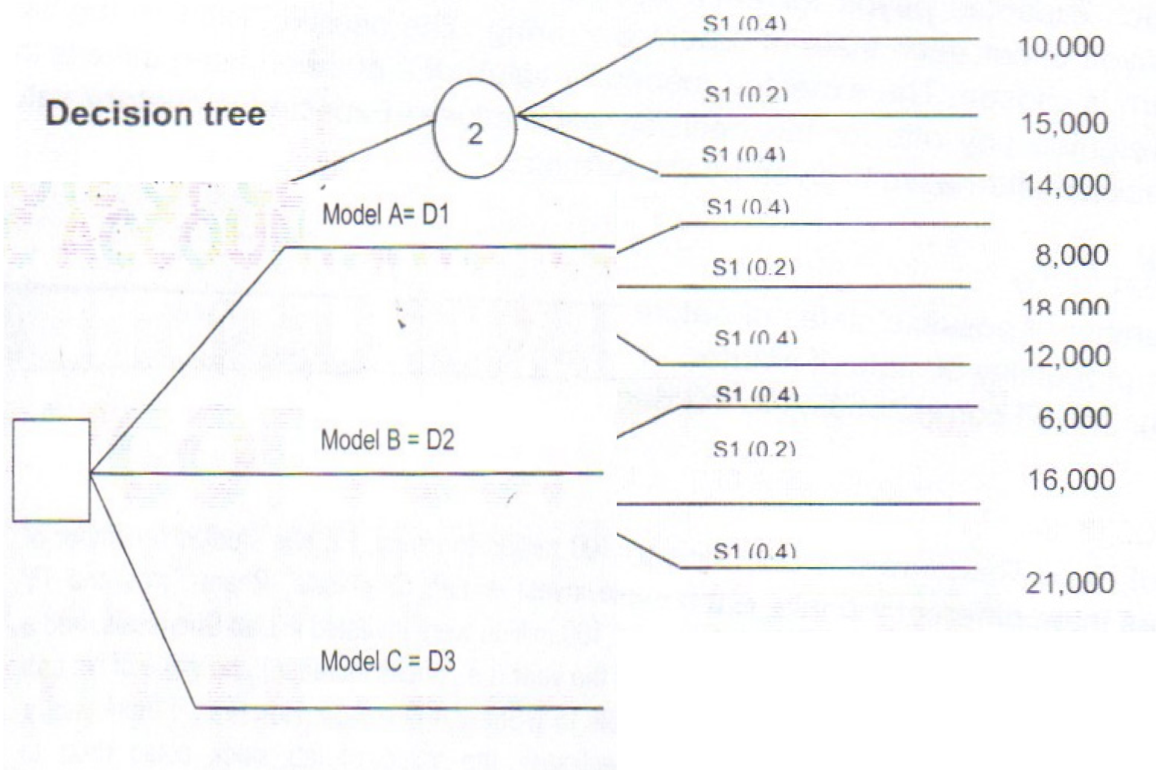
Required:

a) Calculate the expected value for each decision.

b) Advise which model would be most appropriate to open.

Solution:

a) A decision tree can assist in the calculation. Here, let **di, d2, d3** represent decision alternatives of model A, B and C. **S_1**, **S_2** and **S_3** represent states of nature 80,100 and120, respectively.



EMV = 0.4 (10.000) +

0.2. (15.000) + 0.4

(14,000) = 12,600

EMV = 0.4 (8,000) +

0.2 (18,000) + 0.4

(12,000)= 11,600

Model C = D3

- EMV = 0.4 (6,000) +

0.2 (16,000) + 0.4

(21,000)= 14,000

c) Advice choose model C which has the largest EMV.

Example 3

Suppose in this example a developer planning to develop a structure in urban centre obtains data from a research that estimate that the probability of low demand to be 0.35and high demand to be 0.65.

Alternatives	State of nature	
	Low (0.35)	**High(0.65)**
Small	8	8
Medium	5	15
Large	-11	22

Required:

Using the payoff table above determine which alternative should the developer take

Solution:

Alternatives	State of nature		Expected MV
	Low (0.35)	**High (0.65)**	
Small	8	8	8x0.35 + 8x0.65 = 8
Medium	5	15	5(0.35) + 15(0.65)= 11.5
Large	-11	22	-11(0.35)+ 22(0.65) = 10.5

Recall, this is a profit pay off table. The decision should be to build a medium structure because it has the highest expected profit which is the best decision.

Example 4

BAM WE is an investor with Frw 1,100 million to invest. He has studied a number of stocks to invest in and would like to invest in Lab Chemicals, Ream Tyres and TVElectronics. He estimates that if his 1,100 million were invested in Lab Chemicals, and a strong market develops by the end of the year (i.e., prices increase), the value of his Lab Chemicals would be more than double to Frw 2,400 million. However, if there was a weak market (i.e.. stock prices declined), the value of lab stock could drop to Frw 1,000 million Rwandan Francs by the end of the year. His production regarding the value of his Frw 1,100 million for the three stocks in a strong market and weak market are shown in pay off table below.

Purchase	**Strong market (S1)**	**Weak market (S2)**
Lab chemicals	2,400	1,000
Ream tyres	2,200	1,100
TV electronics	1.900	1,150

If the probability for market rise is 0.60, and probability for market fall 0.40.

Required:

Calculate EMV for each decision alternative.

Solution:

Using the payoff table:

Purchase	**Strong markets = S1 = 0.60**	**Weak market S2 = 0.40**	**EMV**
Key chemicals	2400	1,000	1840
Ream tyres	2200	1,100	1760
TV electronics	1900	1,150	1600

i) EMV = 2,400 X (0.60) + 1,000 X (0.40) = 1,840

ii) EMV = 2,200 X 0.60 + 1,100 X 0.40 = 1,760

ili) EMV = 1,900X0.60 + 1,150X0.40= 1,600

On the analysis of expected pay offs the table indicates that purchasing Lab Chemicals would yield the greatest expected profit.

Example 5

Hero Bicycle Company is interested in diversifying into motorcycle manufacturing. The chief executive is not sure whether to start a small plant or a large plant. The market demand is uncertain and will become known only after the plant has been built. If the demand is indeed high and a small plant is built initially then it can be expanded to accommodate high demand. The marketing department has estimated that the probability of high demand is 0.70 and for low demand is 0.30.

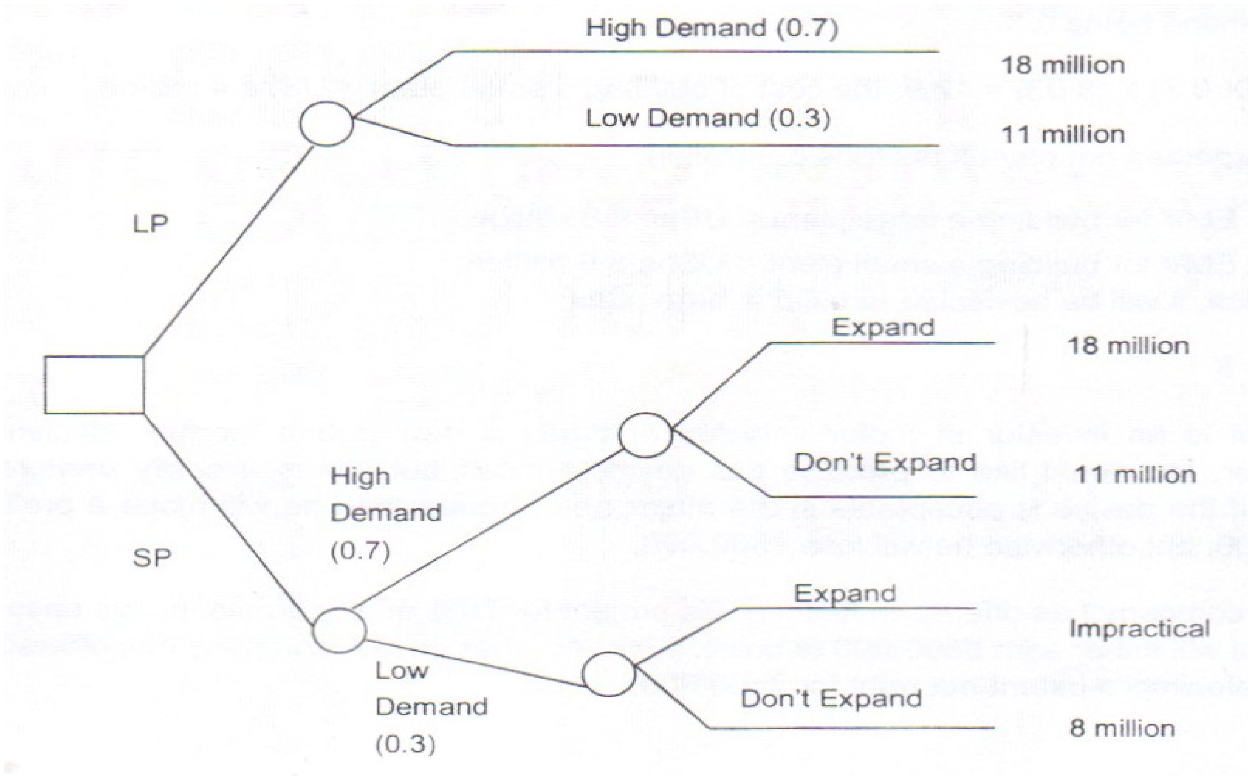
The cost benefit analysis has provided the following information.

Cost of building a large plant - Frw 6 million.
Cost of building a small plant - Frw 4 million.
Cost of expanding a small plant - Frw 3 million.
Revenue for high demand for large plant or small expand plant Frw 18 million.
Revenue for high demand without expansion of small plant Frw 11 million.
Revenue for low demand Frw 8 million.

Required*:*

Determine what should be optimal policy.

This problem can be solved by a decision tree diagram.



Find expected values for each alternative through all branches logically.

For large plant 18 (0.7) x 8 (0.3) = Frw 15.00 million.

Since cost of building the plant is 6 million, net profit = (15-6) = Frw 9.00 million.

Alternatively:

when a small plant is built and the demand is high and the decision to expand is made the revenue = Frw 18 million at of cost expansion of Frw 3 million.

Net revenue for this branch (18-3) =15.

If the demand is high and we no expansion is made the revenue and Frw11 million which is lower than Frw 15 million.

Hence, if the demand is high the decision not to expand would become impractical. When the demand is low, expansion is not logical the revenue of Frw 8 million with probability of low demand being 0.3.

EMV (15 X 0.7) X (8.03) = 12.9, the cost of building a small plant = Frw 4 million.

∆ Hence, expected net pay off is Frw 8.9 million.

The EMV for building a large plant = Frw 9.0 million.

The EMV for building a small plant = Frw 8.9 million.

Hence, it will be advisable to build a large plant

Example 6

Walusimbi is an investor in motor vehicles and has a new patent for fuel efficient carburetor. He would like to produce this device himself but this is a costly venture because if the device is acceptable in the major auto market then he will make a profit $2,000,000, but otherwise he will lose $500,000. A private company has offered to finance this project for 70% of any profits. In this case, Walusimbi will either earn $600,000 or break even. Another private company has offered to buy Walusimbi's patent out right for $100,000.

Required:

What would be Walusimbi's best course of action?

In this case, there are three options available:

Produce the device himself.

Retain 30% interest with device.

Sell the patent.

Net gain depends on whether or not the device is adopted by the major auto market. The decision whether it is adopted or not is beyond the control of Walusimbi.

The net gain for each option under each state of nature will yield expected value EMV. Suppose the Walusimbi estimates the probability of adoption to be 0.25, the probability for no adoption will be 0.75. The expected pay off would be obtained as follows:

E1 = 2,000,000 (0.25) + -500,000 (0.75) = 125,000

E2 = 600,000 (0.25) + 0(0.75) = 170,000

E3 = 100,000 (0.25) + 100,000(0.75) = 100,000

Payoff table

Action	**State of nature**		EMV
	Adopted (0.25)	**Not adopted (0.75)**	
Produce	2,000,000	-500,000	125,000
Retain 30%	600,000	0	150,00
Sell patent	100,000	100,000	100,00

Note: in this case the three decision criteria (maximum, maximax and EMV) lead to three different decisions but Walusimbi would take a decision would yield higher EMV, hence. Walusimbi should retain 30%.

Example 7

The Ndere Troupe is scheduled to carry an outdoor concert on a Sunday afternoon at Nakivubo Stadium. The promoter is worried about the weather on that Sunday that It might rain. Contacts with metrological department forecasters predict that the probability of rain on that Sunday afternoon is 0.24. If it does not rain the promoter is certain to net Frw 1,000,000 but if it rains the promoter estimates to net only Frw 100,000. An insurance company agrees to insure the concert worth Frw 1,000,000 against rain at a premium of Frw 200,000.

***Required**:*

Should the promoter buy the insurance?

Solution:

The promoter has two alternatives of the course of action: insure or not insure.

Payoff table

Alternative	State of nature	
	Rains (0.24)	**Does not rain (0.76)**
Insure	900,000	800,000
Do not insure	100,000	1,000,000

Note: that 900,000 = 1,000,000 - insurance 20,000 + plus 100,000 gate collection.

EMV (insure) = 900,000 x 0.24 + 800,000 x 0.76 = Frw 824,000.

EMV does not insure = 100,000 x 0.24 + 100,000 x 0.76 = Frw 784, 000.

It appears the promoter's best decision is to buy insurance at Frw 200.000 and expect to earn a payoff Frw 824,000.

Example 8

A card manufacturer must decide early enough about the Christmas cards to produce.

She has three possible strategies:

Produce modem cards.

Produce old fashion card.

Produce a mixture of modern and old fashion cards.

Her success depends on the state of the economy in December. If the economy **is** strong, she will do well with modern cards, if the economy is weak old fashion cards will do well.

An in between economy the mixture of both cards will do well. She first prepares a payoff table for all the three possibilities, as below:

Strategies	State of nature		Strong
	Weak	**In between**	
Modern	40	85	120
Old fashion	106	46	83
Mixture	72	90	68

Required:

Suppose the manufacturer reads in a business magazine that the probability of a weak economy is 0.5, a mixture economy 0.2 and a strong economy is 0.3. Determine the EMVand advise on the best strategy.

Solution:

The probabilities provided help to calculate the EMV for each strategy (course of action):

Modern 40(0.5) + 83(0.2) + 120(0.3) = 73

Old fashion 106(0.5) + 46(0.2) + 68 (0.3) = 87.1

Mixture 72(0.5) + 90(0.2) + 68(0.3) = 74.6

Payoff table

Strategy	State of nature			EMV
	Weak (0.5)	**In between (0.2)**	**Strong (0.3)**	
Modern	40	85	120	73.1
Old fashion	106	46	83	87.1
Mixture	72	90	68	74.6

Her strategy should be to produce old fashion and expect a profit is $87,000.

Example 9

Kigali Fire Brigade has trained crews of fire fighters on call to put out any fire any where in the country. The number of calls it receives has been observed to vary between zero and five per month, according to the following distribution.

Number of calls	Probability
0	0.1
1	0.1
2	0.2
3	0.3
4	0.2
5	0.1

The Brigade charges Frw 50,000 for its service. Each crew can handle one call per month and is paid a monthly salary of Frw 10,000 whether the crew is used or not.

Required:

Use the expected value (EMV) to determine the optimal number of crews to keep on call.

Solution:

Its payoff = revenue - cost

If the Brigade has four crew and three calls are received, its payoff is:

= 3 (50,000)-4 (10,000)

= 150,000-40,000

= 110,000

If there are four crews on call and four calls are received, its payoff is:

= 4(50,000)-4 (10,000)

= 160,000

Similarly, other pay off can be calculated.

The payoff table generated is as shown (figures in thousand Rwandan Francs per month):

Action	State of nature					
	Number of calls per month					
Number of crew	**0.1**	**0.1**	**0.2**	**0.3**	**0.2**	**0.1**
	0	**1**	**2**	3	**4**	**5**
1	-10	40	40	40	40	40
2	-20	30	80	80	80	80
3	-30	20	70	120	120	120
4	-40	10	60	110	160	160
5	-50	0	50	100	150	200

Using the payoff table and the given probabilities, calculate EMV for each action:

EMV (1 crew):

-10(0.1) + 40(0.1) + 40 (0.2) + 40 (0.3) + 40 (0.2) + 40 (0.1) = 35

EMV (2 crews):

-20(0.1) + 30(0.1) + 80(0 2) + 80 (0.2) + 80(0.1) = 65

EMV (3 crews):

-20(0.1) + 20(0.1) + 70(0.2) + 120(0.2) + 120(0.1) = 85

EMV (4 crews):

-40 (0.1) + 10(0.1)+ 60 (0.2)+ 110(0.3)+ 160 (0.2) + 160 (0.1) = 90

EMV (5 crews):

-50 (0.1) + 0(0.1) + 50 (0.2) + 100(0.3) + 150 (0.2) + 200(0.1) = 85

Since maximum expected profit is Frw 90,000 when there are four crews on call.

Four crews are its optimal number of crew.

9. Advantages of decision trees

i. They are simple to understand and interpret.

ii. Lay out of events makes alternative courses of action more clear.

10. Disadvantages

i. For data including categories variable with different number level information gained from decision forces can be biased on favour of attributes.

ii. Calculations can be complex particularly if many values of uncertainty or many outcomes are involved.

Self-test questions

Question 1

An oil company, after careful testing and analysis is considering drilling in two different sites. It is estimated that site A will lead to a net profit of $30 million if successful with a probability of 0.2 and a loss of $3 million with a probability of 0.8. Site B will lead to a net profit of $70 million if successful with a probability of 0.1 and a loss of $4 million if not successful with a probability of 0.9.

Required:

Determine which site the company should drill according to the EMV for each site.

Solution:

Site A with EMV = $3.6million.

Question 2

Super college school has sold tickets for music show to be held in the stadium. If it rains the show will have to be moved to the dancing hall which has a much smaller seating capacity.

The college bursar must decide in advance whether to set up the seats and the stage in the dancing hall or in the stadium or both just in case. The payoff table shows the net profit in each case.

Strategies	**States of nature**	
	Rain	**No rain**
Set up his stadium	-1,550	1,500
Set up in dancing hall	1,000	1,000
Set up both	750	1,400

(Figures in Frw '000)

Required*:*

If the weather forecast predicts rain on the show day with a probability 0.6. what strategy should the bursar choose to maximise expected profit.

Solution:

Set up both and expect EMV Frw 1,010.000.

Question 3

Suppose that you must choose between two stocks (a1) and (a2) and there are only two states of nature (θ_1 *and* θ_2) and having the same probabilities (P(θ_1) = *P(*θ_2*)* = 0.5.

Required:

Given the following data in the table.

	States of nature	
	θ_1	θ_2
Actions	a1= Frw -50	Frw 100
	a2 = Frw 10,500	Frw -10,000
Probability	0.5	0.5

Units in Frw 1,000's.

Determine the stock that should be acquired.

Solution:

Stock a2 and expect EMV Frw 250,000.

Question4

A company that deals in bottling drinking water is considering adding a new water boiler to its factory in an attempt to avoid costly delays in case one the present boilers shut down for repairs. The company has determined that the following payoff table is appropriate for its present projection in five years.

		State of nature		
		No repairs	**Minor repairs**	**Major repairs**
Actions	Do not add new boiler	0	-4,000	-15,000
	Add new boiler	-10,000	-10,000	-10,000
	Probability	0.2	0.3	0.5

Units m (Frw 000).

Required:

Find the optimal action that the company should take using EMV approach.

Solution:

Should not buy attend to minor repairs at a cost of Frw 4,000,000 and major repairs at Frw 15,000,000 and expect a loss of Frw 8,700,00 compared to buying and still incur a loss of Frw 10,000,000.

Question 5

An Estate Management Agency (EMA) is considering two levels of investment in real – estate development; a low participation (A1) and a high participation (A2). Two states of nature are deemed possible, a partial success {B1) or a complete success (B2). The payoff matrix is estimated to be:

	B1	**B2**
A1	-200	400
A2	-500	1,000

Required:

How large does the prior probability of B1 have to be in order to make action A1 the better choice.

Solution:

Any value of P(B1) $>\frac{22}{33}$ will make A1 best action.

Question 6

A plastic company has had adhesion problems in its experiments with chrome plating on plastic butterfly valve because of irregularities in electricity flow during plating process. On the basis of past data, management knows approximately 70% of the time the current is fairly uniform; in which case 90% of each batch of 1000 valves produced will be good and only 10% will be defective. The other 30% of the time, when current is somewhat irregular only 60% of the valves are good and 40% are defective. Unfortunately, engineers cannot determine how good the current will be in testing process. All they know is 90% or 60% of the valves in each batch will be good. The company has several alternative ways to handle each batch.

One alternative is to send the batch directly to the next operation assembly and hope for the best; their records shows that when they do this they would incur costs of delay and adjustment of about 1,000 for each batch in which 90% of the valves are good and 4,000 in which 60% are good. Another alternative is to rework the entire batch. This process will ensure that the batch is sufficiently from defects that no delay and adjustment costs occur, however, reworking will cost 2,000.

The relevant data are shown in the payoff table below.

		States of nature	
		90% good	**60% good**
Actions	Assembly	-1.000	-4,000
	Rework	-2,000	-2,000
	Probability	0.7	0.3

Units in (Frw '000).

Required:

Using a payoff matrix table or decision tree, find the optimal solution for plastic company.

Solution:

The optimal solution is to send directly and incur EMV = cost Frw 1,900.000 compared to rework and EMV = cost Frw 2,000.000.

Question 7

HK restaurant has Frw 100,000 available to prepare either tea or coffee in preparation for a function that takes place annually in the location. A decision must be made early for planning. However, the restaurant is unable to predict whether the day will be cold or hot. If tea prepared and the day is not cold, the payoff will be Frw 120,000; but if the weather is cold the payoff will only be Frw 105,000. If coffee is prepared, the pay offs for hot and cold weather are Frw 110,000 and Frw 125,000, respectively. Past records show that 70% of similar functions were hot and 30% were cold. *(Adapted from the Public Accountants ExaminationsBoard, 2012.)*

Required:

Determine the two expected pay offs and arrive at the decision.

Solution:

EMV for tea = Frw 115,000 and EMV for coffee = Frw 114,500. Decision - prepare tea.

Question 8

KIDA Communications sells units of communication equipment to both the military and civilian markets. The following year's sales depend on market conditions that cannot be predicted exactly. The company follows the modern practice of using probability estimates of sales based on the informed opinion of the company's executive. The military division estimates its sales as follows:

Units sold	1,000	3,000	5,000	10,000
Probability	0.1	0.3	0.4	0.2

The corresponding sales estimates for the civilian division is as follows:

Units sold	300	500	750
Probability	0.4	0.5	0.1

Taking x to be the number of military units and y to be the number of the civilian units, and that the company makes a profit of Frw 2,000 on each military unit and Frw 3,500 on each civilian unit, the following year's profit accrued is given by Z = 2,000x + 3,500y.

Required:

Compute the expected:

i) Value for the military;
ii) Value for civilian; and
iii) Profit.

Solution:

i) E(V) for military = 5,000.
ii) E(V) for civilian = 445.
iii) Profit = Frw 11,557,500.

Question 9

An apple trader has established that his daily sales follow the following probability distribution:

Sales	100	200	300	500
Probability	0.2	0.3	0.4	0.1

The cost for each apple is Frw 500 and the trader's retail price is Frw 1,000 per apple. At the end of each day, the remaining stock is cleared at Frw 250 each.

Required:

Calculate the expected profit of the trader when he stocks 400 apples.

Solution:

Expected profit = Frw 80,000.

Questions on Game theory

Q: What is game theory?

A: The study of decision making under competition.

Q: How is that different from payoff tables?

A: With a payoff tables (called decision theory) there is only one player (the decision maker) and columns represent randomly occurring states-of-nature. With game theory, there are two players, so the columns of the decision table will represent the intelligent actions of a competitor.

Q: How do you set up a table for game theory?

A: The rows represent your strategies (alternatives) while the columns represent the strategies (alternatives) of your competitor. The numbers in the cells of the table represent your costs (or gains) based on the two strategies chosen (yours and your competitor's).

Q: Are you concerned about the payoffs for your competitor?

A: That depends on whether you have a non-zero-sum game or a zero-sum game.

Q: What is a non-zero-sum game?

A: A competition where the amount won or lost by one player does not directly reflect the amount lost or won by the other player.

Q: Could you give an example of that?

A: The most famous example is the Prisoner's Dilemma:

D: Two criminals have been arrested for a crime. They really did commit the crime, but the police have insufficient evidence to get a conviction. Therefore, the police put the prisoner's in separate cells and give each a chance to confess. Thus, each criminal has two options (or strategies, or alternatives): don't confess or confess. The table for this situation looks like Table 1:

		Prisoner 2	
		Not Confess	Confess
Prisoner 1	Not confess	5, 5	0, 10
	Confess	10, 0	1, 1

Table 1: Prisoner's Dilemma

The payoffs show the "utility" to each prisoner (prisoner 1, prisoner 2). If neither confesses, then they go free and get to split the proceeds of their crime (they each get a payoff of 5). If one confesses and the other does not, then the police let the one that confesses go free (utility of 10) while the other goes to jail (utility of zero). If both confess, then they both go to jail for a reduced term (utility of 1, each). The question is: what should you, as one of the prisoners, do?

Q: What if I decide not to confess?

A: Then you run the risk that your partner will confess, sticking you with a long jail term while s/he goes free.

Q: What if I decide to confess?

A: That's the reasonable thing to do, because no matter what your partner decides, you come out ahead. If your partner doesn't confess, then a utility of 10 is better than 5, but if s/he does confess, a utility of 1 is better than a utility of 0.

Q: So I should confess?

A: Yes, but realize that your partner will make the exact same decision.

Q: So we will both confess?

A: That is the logical thing to do, but notice: that means you will both go to jail and receive a utility of 1, whereas if you had cooperated (neither confessed) you could both have gone free and had a utility of 5.

D: This is an example of a general problem in large societies: it is almost impossible to get people to act cooperatively, even though they would all be better off if they did. As an example, look around the next time you are in a traffic jam. The traffic would flow much more smoothly if everyone would slow down and let each other change lanes, but as soon as *you* do that, someone takes advantage of you, so you get angry and start being aggressive.

Q: Why did you say this was a problem for "large societies?"

A: In small societies, public opinion is, usually, a sufficient goad to enforce cooperative behavior.

Q: So, what is the solution to this dilemma?

A: There isn't one. This is, interestingly enough, the rationale for the liberal political viewpoint. Since people won't cooperate, force them to do so by passing laws. Unfortunately, good laws are hard to write (try to write a law to accomplish some worthwhile goal that has no loopholes at all), so governments rarely accomplish their goals (which is the rationale for the conservative political viewpoint).

Q: If all this is true, then why do some people cooperate?

A: That's a really good question and there are reams of research on that point. The answer is – we don't know. For myself, I cooperate when I view the penalty (zero utility if someone takes advantage of me) as acceptable – as in a traffic jam, where I tend to hang back and let people cut me off.

D: From a philosophical viewpoint, this becomes a discussion about morality. Religions try to coerce cooperative behavior (calling it "moral" or "good" behavior) by increasing the penalty for non-cooperative behavior (some sort of punishment in an afterlife). Whether or not you believe in a religion is not relevant – what is fascinating is the attempt to create some non-religious basis for cooperative (or moral, or good) behavior. They have been trying this since Ancient Greece, and have not yet attained it. Philosophers have spent centuries pointing out that non-cooperative behavior is ultimately damaging to both the individual and the society (or to put it another way, hedonism is no fun), but very few people are philosophers.

I wish we had time to go into this dilemma, because it underlies all political and most economic theory, and this is one example you can see playing out all around you every day. For you, knowing about this may help you do a better job of anticipating what other people will do, which will help you make good decisions.

Q: Is a zero-sum game much the same thing?

A: Not at all. A zero-sum game exists when your gains (for a given pair of strategies) is exactly the same as your opponent's losses.

Q: Do these games lead to a different dilemma?

A: No, there are actually many times when rational thinking will lead both players to a win-win solution.

Q: Are there really situations like that?

A: Yes, contract negotiations, military conflict, political struggles, or simply competition for a market.

Q: Can you give an example?

A: Pretend you are an athlete, negotiating a contract with a team. You have two basic strategies: sign the contract offered or holdout, while the team has a variety of options (combining length of contract, bonuses, clauses, …). In this example, the team is restricting itself to three contract strategies, called A, B and C. This gives us Table 2:

	A	B	C
Sign	$50,000	$35,000	$30,000
Holdout	$60,000	$40,000	$20,000

Table 2: Non-zero-sum contract negotiations

Q: How do you analyze Table 2?

A: If the game is to be played once (each player makes a single decision, simultaneously), then the best approach is a maximin strategy.

Q: Is that something like the minimax regret rule we used in payoff tables?

A: Very similar, except we are not calculating regret.

Q: How is a maximin strategy applied?

A: For each strategy, find the minimum payoff, then choose the largest (maximum) of those minimums.

Q: Why should I choose the minimum payoff? Wouldn't I prefer the maximum?

A: You would prefer the maximum, but you are playing against an opponent who will make his/her decision trying to offset whatever you do. Your opponent's strategy, by the way, is a minimax rule, since s/he is working against you.

D: Consider Table 2. The highest payoffs for you (as the athlete) are from holding out, but the team negotiator knows that as well as you do. Anticipating that you will decide to hold out, the negotiator will choose Contract C, which reduces your payoff to $20,000. That is less than the $30,000 you could have made if you had not held out. Therefore, you anticipate that the negotiator will make the rational decision for Contract C, so you make the rational decision to sign. The only reason to hold out is if you think the negotiator will make a mistake, which isn't rational. Interestingly, if your opponent assumes you will choose to sign, s/he would still choose Contract C.

Q: Do all non-zero-sum games work out like this?

A: No. When they do, this is called an equilibrium point, because it is the minimum of a row (which your opponent wants) and the maximum for a column (which you want). When they don't, they are referred to as a "mixed-strategy" game, and the analysis breaks down.

Q: Can you give an example of a mixed-strategy game?

A: Two politicians are campaigning against each other. There are two days left until the election, and the politicians are planning where to campaign. There are two major cities in their district: Newtown and Oldburg. Thus, each politician has three strategies:

Strategy 1: spend 1 day in each city
Strategy 2: spend both days in Newtown
Strategy 3: spend both days in Oldburg

The payoffs are the total net votes taken from the opponent, expressed in 000's of votes, so the payoffs for Politician 2 are simply the reverse of the payoffs for Politician 1 shown in Table 3:

			Politician 2	
		1 day in each	Both in Newtown	Both in Oldburg
	1 day in each	0	-2	2
Politician 1	Both in Newtown	5	4	-3
	Both in Oldburg	2	3	-4

Table 3: Non-zero-sum mixed strategy table

Q: What should Politician 1 do?

A: If Politician 1 tries to minimize his losses (maximin rule), s/he would look at the minimum payoff for each strategy (-2, -3, and -4, respectively) and choose to spend one day in each town.

Q: What would Politician 2 do?

A: For Politician 2, a high payoff is a bad outcome, so s/he looks at the highest number for each column and chooses the smallest (minimax rule). Doing that, the maximums are 5, 4, and 2, respectively, so Politician 2 would choose to spend both days in Oldburg.

Q: Doesn't that result in Politician 1 gaining 2,000 votes?

A: Yes; this tells Politician 2 that s/he should not use the minimax rule. Instead, s/he should anticipate that Politician 1 will choose to spend one day in each city, and Politician 2 can actually do much better, gaining 2,000 votes instead of losing 2,000 votes, by choosing to spend both days in Newtown.

Q: Can Politician 1 anticipate this?

A: Since both are being rational, Politician 1 would indeed anticipate Politician 2 choosing to spend both days in Newtown, which means s/he would look at column 2 and decide to also spend both days in Newtown, thus gaining 4,000 votes rather than losing 2,000 votes.

Q: Is that where it ends?

A: No; Politician 2 would once again anticipate Politician 1 switching to spending both days in Newtown, and would change plans to spend both days in Oldburg, to gain 3,000 votes rather than lose 4,000.

Q: Isn't that where we started?

A: Exactly. This sort of game cannot be solved by any sort of mixed strategy. It cannot even be solved by trying to out-wait your opponent, since the opponent can always switch strategies whenever you make your decision.

Q: What should you do?

A: One theory says to flip a coin – doing something random prevents your opponent from anticipating what you will do. Another says to estimate probabilities that your opponent will choose each strategy (with, of course, no indication as to *how* you are supposed to do that), and calculate a weighted average (or expected) payoff for each strategy. There are graphical solutions, and algorithmic (constrained optimization) solutions, but none of them are very satisfactory.

Q: What do we learn from game theory?

A: If you were looking for answers for a single problem, you'll be disappointed. If, however, you are going up against the same opponent over and over again, then you can keep track of the choices your opponent made each time, and begin to learn to anticipate how s/he makes decisions. That can give you an advantage.

CHAP 26.

LINEAR PROGRAMMING

26.1. Study objectives

By the end of this chapter, you should be able to:

define the term linear programming;

discuss the role of linear programming in management problems;

explain the applications of linear programming in management;

state the assumptions of linear programming;

state the advantages of linear programming;

describe the limitations of linear programming;

define the terminologies used in linear programming - objective function, constraints, optimum solution, feasible region;

explain the procedure in the formulation of linear programming models;

explain the graphical method of linear programming and give its limitations;

formulate linear inequalities from the given constraints, represent them by graphical method, and identify the feasible region;

use the graph obtained to find the optimum solution, and the value of the objective function;

identify the standard maximisation problem;

use the simplex method to solve maximisation LP problems: and

Solve the LP problems for mixed constraints.

Definition of Linear Programming

Linear programming is the mathematical technique that deals with finding the best solution for the problems of maximising or minimising a linear function called the objective function subjected to constraints expressed by linear inequalities.

The largest or smallest value of the objective function is called the optimal value and a collection of values of x, y, z that give the optimal value give the optimal solution.

The variables x, y, z are called the decision variables.

a) Definitions of the terminologies used in linear programming

Objective function

This is the linear function to be either maximised or minimised given the constraints.

The objective function may be:

i) a profit, revenue or contribution function that needs maximising subject to a set of constraints; *

ii) a cost function that needs to be minimised subject to a set of constraints.

Constraints

These are limitations or restrictions expressed as a system of linear inequalities.

Feasible region

This is the region on the x-y plane which contains all the integral points that satisfy the given constraints.

Optimum solution

This is the best solution from the feasible region that either maximises or minimizes the objective function.

Linear programming problem

This is a problem consisting of an objective function subjected to constraints expressed as linear inequalities.

b) Assumptions of linear programming

Linearity constraints

Linearity constraints are assumed to be linear.

Proportionality

This means that each decision variable in every equation must appear with a coefficient, that is, the decision variable must only be multiplied or divided by a number.

Additive

This means that the decision variables are added or subtracted together never to be multiplied or divided by each other, that is, in the objective function additive means that the contribution of the variables to the objective is assumed to be the sum of their individual weighted contributions.

Divisibility

The decision variables can also take on fractional values in order to make a LP model computationally tractable, one must be prepared to accept non-integer solutions.

Certainty

This means that the LPP is assumed to have no probable elements. This is unrealistic, in the real world, some degree of uncertainty is always present.

26. 2. Advantages of linear programming

LP improves the quality of decisions. The manager who uses linear programming methods becomes more objective than subjective.

LP helps in providing better tools for adjustment to meet changing conditions.

LP helps in attaining the optimum use of productive factors.

LP can handle business problems with constraints like market demand, availability of raw materials.

26.3. Limitations of linear programming

The objective function and the constraints in some problems are not linear. LPP under non-linear condition usually results in a wrong solution.

The parameters in the LP model are assumed to be constant. But in real life situation they are neither constant nor deterministic.

LP is applicable to only static situations since it does not take into account the effect of time.

LPP may get fractional valued answers for the decision variables, however only the integer values of the variables are rational and logical.

LPP deals with problems that have a single objective. Some real-life problems may have more than one objective.

LP cannot be used effectively for large scale problems i.e. those that may have more than two decision variables, this leads to enormous computational challenges.

26.4. The role of linear programming in solving management problems:

Managers use LP to determine the best way to use the available resources. They use LP to make decisions about the most efficient use of limited resources such as money, materials, machinery, time, etc.

26.5. Procedure for graphical method of linear programming method

In order to sketch the region represented by a linear inequality in two variables, the following steps are taken:

Step 1 : Restrictions. There are always two non-negativities, that is $x \geq 0$ and $y \geq 0$.

Step 2: Expressing the sets of constraints as linear inequalities.

Step 3: Writing down the boundary line should be a continuous (solid) if the inequality is either r$\geq$ or$\leq$. The boundary line should be dashed or broken if the inequality is either < or >

Step 4: Find the x and y intercepts for each of the line.

Step 5 : Plot the boundary line shading off the unwanted region, corresponding to each of the given inequalities

Step 6: Repeat steps 2, 3 and 4if more than one inequality is given.

Step 7: Choose a test point not on the line and substitute it in the given inequality.

Step 8: If the test point satisfies the inequality, then all the points within that region are wanted, so we shade off the region in the other side of the line.

Example 26.**1**

A small-scale industry manufactures electrical regulators the assembly of which is being accomplished by a small group of skilled workers both men and women. Due to the limitations of space and finance, the number of workers employed cannot exceed eleven and their salary bill not more than Frw 6,000,000 per month. The male members of the skilled workers are paid Frw 600,000 per month while the females doing the same work get Frw 500,000 per month. Data collected on the performance of these workers indicate that a male member contributes Frw 1,000,000 per month to the total returns of the industry while the female member contributes Frw 850,000 per month.

Required:

Solve the problem graphically and determine the number of male and female workers to contribute maximum revenue.

Solution:

Step 1: Let the decision variables be x and y, where x is the number of male workers to be employed and y is the number of female workers

Step 2: Summarise the given information in table form:

Constraints	Male	Female	Total
1. Number of workers	X	Y	Does not exceed 11
2. Salary per month	600,000	500,000	Not more than 6,000,000

3. Returns per month	1,000,000 850,000

Step 3: Expressing the constraints as linear inequalities:

i) Non-negativity x $\geq$ 0 and y $\geq$ 0

ii) x + y<11

iii) 600,000x + 500,000y $\leq$ 6,000,000

6x + 5y $\leq$60 (dividing through by 100.000)

Objective function R = 1,000,000x + 850,000y

Step 4: Writing boundary lines.

For x $\geq$ 0, boundary line is x = 0(solid line)

For y $\geq$ 0, boundary line is y = 0(solid line)

For x + y 5 11, boundary line is x + y = 11 (solid line)

For 6x + 5y $\leq$ 60.boundary line is 6x + 5y = 60(solid line)

Step 5: Finding the x and y intercepts for each of the boundary lines:

For x+y=11

If	X	0	11
then	y	11	0

For 6x+5y=60

If	X	0	10
Then	y	12	0

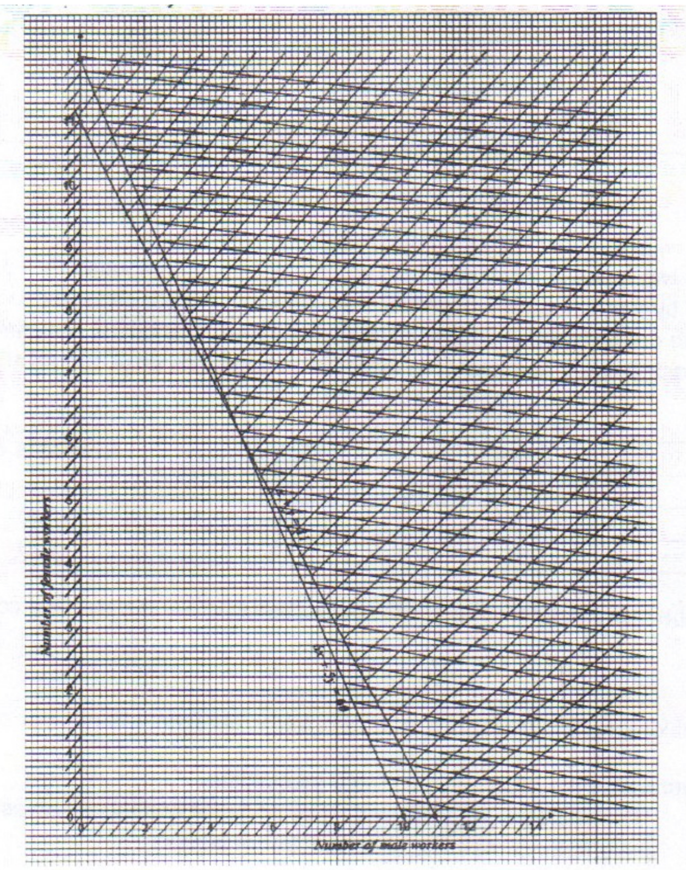
Step 8: Choose an appropriate scale covering the highest value of x and y obtained in the tables above, that is, for x the highest value is 11, whereas y is 12, draw the grid with labelled axes.

Step 9: Draw one boundary line at ago, by using the points in the table for example for line x + y = 11 points to plot are (0,11) and (11,0), being mindful of its nature whether solid or dashed.

After drawing the line, label it as x + y =11.

Step 10: Before drawing the next line, choose a test point not on the line and substitute it in the original inequality of the boundary line. The best point to choose is (0, 0). Is 0 + 0 $\leq$11 ? Yes, then (0, 0) is in the wanted region, shade off the unwanted region which is above the line x + y = 11.

Stept1: Repeat steps 9 and 10 for the next boundary line.



Step 12: Read the corner points of the feasible region.
Step 13: Substitute the corner points in the objective function and determine the optimum solution.
Note: The steps seem to be many for the sake of explaining to you but when you master the basics they are reduced to about only five.
For maximum returns:

Corner	Objective function R = 1,000,000x + 850,000y
(0,11) (5,6) (10,0)	9,350,000 10,100,000 10,000,000

Five males and six females would contribute maximum total returns of the industry

Example 26**. 2**

A firm makes two types of furniture chairs and tables. The contribution for each product as calculated by the accounting department is Frw 20,000 per chair and Frw 30,000 per table. Both products are produced on three machines A, B and C. The time required in hours by each product and total time available in hours per week are as shown in the table below.

Machine	Chair	Table	Available time
A	6	6	72
B	10	4	100
C	4	12	120

How should the manufacturer schedule his production in order to maximise contribution? **Solution:**

Step 1: Let x represent the number of chairs. Let y represent the number of tables.

Step 2: Expressing the constraints as linear inequalities:

- i) Non-negativity x > 0 (cannot make negative number of tables)
- ii) y > 0 (cannot make negative number of tables)
- iii) 6x + 6y < 72
  x + y < 12 (dividing through by 6)
- iv) 10x + 4y<100
  5x + 2y S 50(dividing through by 2)
- v) 4x + 12ysi20
  x + 3y < 30(dividing through by 4)

Step 3: Finding the x and y intercepts for each of the boundary lines:

Example 2

A firm makes two types of furniture chairs and tables. The contribution for each product as calculated by the accounting department is UShs 20,000 per chair and UShs 30,000 per table. Both products are produced on three machines A, B and C. The time required in hours by each product and total time available in hours per week are as shown in the table below.

Machine	Chair	Table	Available time
A	6	6	72
B	10	4	100
C	4	12	120

How should the manufacturer schedule his production in order to maximise contribution?

Solution:

Step 1: Let x represent the number of chairs. Let y represent the number of tables.

Step 2: Expressing the constraints as linear inequalities:

- i) Non-negativity $x \geq 0$ (cannot make negative number of tables)
- ii) $y \geq 0$ (cannot make negative number of tables)
- iii) $6x + 6y \leq 72$
  $x + y \leq 12$ (dividing through by 6)
- iv) $10x + 4y \leq 100$
  $5x + 2y \leq 50$(dividing through by 2)
- v) $4x + 12y \leq 120$
  $x + 3y \leq 30$(dividing through by 4)

Step 3: Finding the x and y intercepts for each of the boundary lines:

For $x + y \leq 12$,
boundary line is $x + y = 12$ (solid line)

If	x	0	12
Then	y	12	0

Points to plot (0,12) and (12, 0)

For $x + 3y \leq 30$,
boundary line is $x + 3y = 30$ (solid line)

If	X	0	30
Then	Y	10	0

Points to plot (0,10) and (30, 0)

For $5x + 2y \leq 50$,
The boundary line is $5x + 2y = 50$

If	x	0	10
Then	y	25	0

Points to plot (0, 25) and (10, 0)

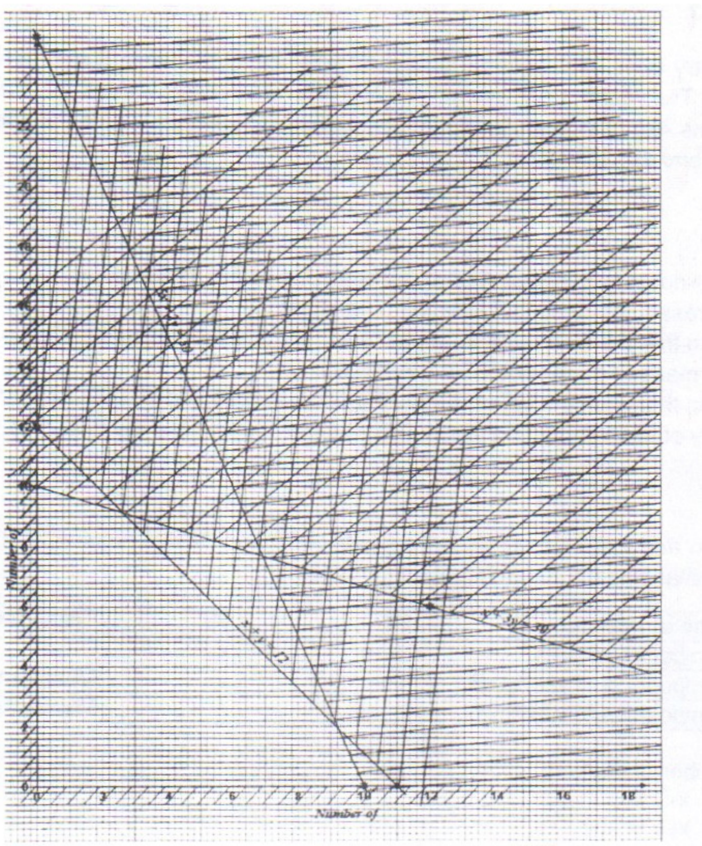
Step 4:" Draw each line, label it and shade off the unwanted region. Choose a test point either above or below the drawn line, say (0, 0) and substitute it in the original inequality:

For x + y≤12 is 0 + 0 ≤12 ? Yes, then shade the region above the line x + y = 12

For 5x + 2y≤ 50 Is 0 + 0 ≤ 50? Yes, then shade the region above the line 5x + 2y = 50

For x + 3y ≤30 Is 0 + 0 ≤30? Yes, then shade the region above the line x + 3y = 30

Step 5:



Step 6: Corner points within the feasible region are entered in the table:

Corner point	Contribution function maxZ = 20,000x + 30,000y
(0,0)	Z = 20,000 x 0 + 30,000 x 0 =0
(10.0)	Z = 20,000 x 10 + 30,000 x 0 = 200,000
(3,9)	Z = 20,000 x 3 + 30,000 x 9 = 330,000
(0,10)	Z = 20.000 x 0 + 30,000 x 10 = 300,000

The manufacturer would produce three chairs and nine tables in order to realise maximum contribution

Example 26.3

On a poultry farm, birds require to have at least 20kg of vitamin B and at least 28kg of vitamin C. The vitamins are available in two different packets P1 and P2. Each packet of P1 contains 4kg of vitamin B and 2kg of vitamin C. Each packet of P2 contains 2kg of vitamin B and 4kg of vitamin C.

Required:

a) Write down four inequalities to represent the above information.

b) Represent the above information on a graph.

c) Given that the total number of packets of P1 and P2 combined is to be a minimum, how many of each should be taken daily?

d) Given that each packet of P2 and P1 costs 25,000 and 10,000, respectively, how many of each should be taken daily in order to minimise the total cost.

Solution:

a) Let xi and **X2** be the number of packets of P1 and P2 due required daily.

Summarising the information in tabular form:

Type of vitamin	**P1**	**P2**	Total mass
Number of packets	Xi	x_2	
Composition of vitamin B	4	2	Minimum of 20
Composition of vitamin C	2	4	Minimum of 28

Number of packets

i) $x_1 \geq 0$

ii) X**2** $\geq$ 0

iii) $4x_1 + 2x_2 \geq 20$

iv) $2x_1 + 4x_2 \geq 28$

b) Finding the boundary lines and indicating whether solid or dotted line:

$X_1 = 0$ (solid line)

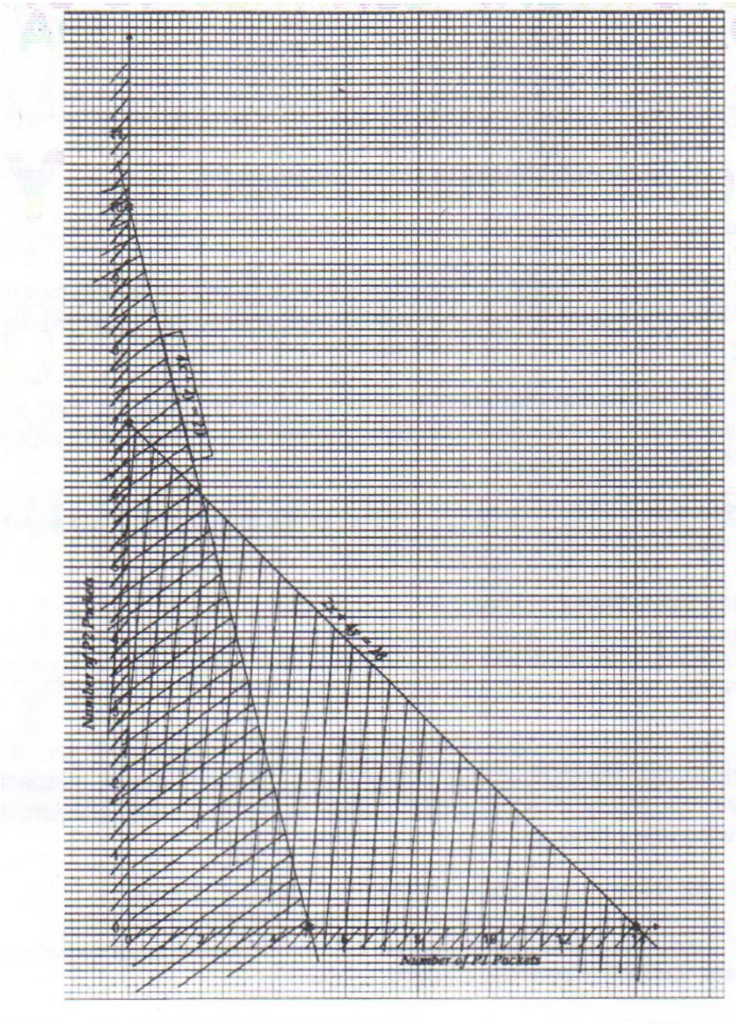
$X_2 = 0$ (solid line)

$4x_1 + 2x_2 = 20$(solid line) = $2x_1 + x_2 = 10$

$2x_1 + 4x_2 = 28$(solid line) = $\mathbf{X_1 +} 2x_2 = 14$

c) From the graph two packets of Pi and six packets of P2 should be taken daily.

a) Minimum cost = (25,000x6) + (10,000x2) = Frw170,000/=



7. Advantage of the graphical method

It is relatively easier to interpret and apply.

8. Disadvantages of the graphical method

Restricted to problems with only two variables.

Generates a feasible solution from which the optimum solution has to be obtained through investigation.

Gives rise to the optimum solution alone and when need be the shadow prices or costs have to be determined separately.

26.6. Simplex method for maximising in linear programming

Definition

A simplex method of solving an LP problem is a progressive method from a position of zero production and therefore zeros contribution, until no further contribution can be made

Each step produces a feasible solution and each step produces an answer better than the one before, that is, either greater contribution in a maximising problem, or less cost in minimising problem.

Here we shall limit ourselves to describing how to use the technique but avoid trying to explain the complex mathematics behind the simplex method.

Formulating the simplex model:

First, the problem is stated in the general format, that is, the objective function and the constraints.

Secondly, the inequalities are converted to equations. For instance, a tailor produces skirts and trousers, which take two and five hours to sew, respectively, and he has 120 hours to accomplish a given order. In a standard format the constraint would be written as:

$2x_i + 5x_2 + s = 120$, where x_i= number of skirts, x_2= number of trousers.

The above constraint is converted into an equation by adding an extra variable called a slack variable to give the equation below;

$2x_i + 5X_2 + s = 120$

The slack variable, s represents any unused capacity in the constraint. It can therefore take any value from 120 hours (i.e., the position of zero output and therefore maximising the unused capacity), to 0 hours (i.e., the position of the time being fully utilised and. therefore, 0 unused capacity).

Each constraint will have its own slack variable, and once the slack variable has been included into the constraint the simplex method automatically assigns it an appropriate value at each iteration.

A simplex maximising example

Example 26.**4**

Given the following standard linear programming maximising problem,

Objective function:

Maximise, $Z = 3x_1 + 5x_2 + 2x_3$

Constraints:

Subject to $5xi + 3x_2 + x_3 < 300$

$X_1 + 2x_2 + \mathbf{X_3} \leq 80$

$5x_1 + 6x_3 \leq 100$

$3x_1 + x_2 \leq 45$

$X_1 \geq 0$, $x_2 \geq 0$, $X3 \geq 0$

Required:

The problem above is to be solved by the simplex method, set the initial tableau for the solution.

Solution:

Changing the constraints to equations, the problem becomes:

Maximise, $Z = 13x_1 + 5x_2 + 22x_3$

Subject to $5x_1 + 3x_2 + \mathbf{X_3} + S_1 = 300$

$\mathbf{X_1} + 2X_2 + X_3 + \mathbf{S_2} = 80$

$5x_1 + 6x_3 + s_3 = 100$

$3x_1 + \mathbf{X_2} + \mathbf{S_4} = 45$

Where S_i, S_2, $\mathbf{S_3}$ and s_4 are slack variables.

Therefore, the initial tableau for the problem above is given by:

	variables			slack variables				
Solution Variable	**X1**	**x_2**	**x_3**	**S_1**	**S_2**	**S_3**	**S_4**	**solution**
S_1	5	3	1	1	0	0	0	300

s_2	1	2	1	0	1	0	0	80
S_3	5	0	6	0	0	1	0	100
S_4	3	1	0	0	0	0	1	45
Z	13	5	22	0	0	0	0	0

Example 26. **5**

Trinity Clays produces three types of bricks A, B and C. The profit earned from each type of brick, in millions is 8, 5 and 10, respectively per week. The machine used can work for 400 hours in a week and takes the machine 2, 3and 1 hour(s) to make brick A. B and C, respectively. There are only 150 units available in a week of a special material for decorating bricks A and C. 200kg only of a setting chemical is to be used in this period of which brick A takes2kg per brick while brick B takes 4kg per brick. It is Trinity Clays policy not to make more than 50 bricks of type B in a week.

Required:

a) Write down:

i) the objective function;

ii) all the constraints; and

iii) set up the initial tableau for the problem.

b) Hence, establish the production plan which maximizes Trinity Clays weekly profits.

Solution:

Let x_1, $X_2$and X_3be the number of bricks of type A, B and C produced per week by Trinity Clays per week.

a) i) Objective function: maximise profit, $Z = 8x_1 + 5x_2 + 10x_3$

ii) Constraints : $2x_1 + 3x_2 + x_3 < 400$

$x_1 + x_3 \leq 150$

$2x_1 + 4X_3 \leq 200$

$x_2 \leq 50$

$X_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

iii) Introducing the slack variables:

maximise, p = 8x1 + 5x2 + 10x3

subject to: **2xi** + 3x2 **+** x_3 **+ Si** = 400

Xi + X3 + S2 = 150

2xi + $4x_3 + s_3$ = 200

X2 +s_4 = 50

The initial tableau:

		Variables		Slack variables				
Solution								
Variable	X_1	X_2	x_3	**Si**	s_2	s_3	S_4	**Solution**
S_1	2	3	1	1	0	0	0	. 400
S_2	1	0	1	0	1	0	0	150
S_3	2	0	4	|0	0	1	0	200
S_4	0	2	0	0	0	0	1	50
Z	8	5	10	0	0	0	0	0

b) The initial tableau represents a feasible solution. However, it is at no production and therefore the slack variables are at maximum values (i.e., **Si**=400, s_2 =150, s_3 = 200 and s_4 = 50). This gives zero profit (i.e., Z = 0). The improvement process is done gradually as follows:

i) Identification of the pivot element:

Select the highest contribution in z-row, that is, 10 under **X3.**

Divide the positive numbers in the x3 column into the solution column. That is, 400 ÷ 1 = 400, 140 ÷1= 150,200 ÷ 4 = 50, 50 ÷ 0 = 0 ignore.

50 being the smallest result, 4 marked by a box (see the initial tableau) is identified as the pivot element, that is, under x3 and along the row with the smallest result after the division.

All elements in the row along the pivot element are divided by the pivot element to give new elements in the improved tableau for the same row.

The rest of the rows are adjusted using row operations on the new row along the pivot element also to give corresponding rows in new tableau and must give zero values under x3.

Therefore:

2nd Tableau:

		Variables			**Slack variables**				
Row	Solution variable	X_1	X_2	x_3	S_1	S_2	S_3	s_4	Solution
1	**Si**	2	3	1	1	0	0	0	400
2	s_2	1	0	1	0	1	0	0	150
3(row s_3÷4)	x_3	$^1/_2$	0	1	0	0	$^1/_4$	0	50
4	s_4	0	**1**	0	0	0	0	1	50
5	**z**	8	5	10	0	0	0	0	0

There being no positive values in the Z row is an indication that the optimum solution has been achieved and therefore the 5th tableau is the final one in the simplex method solution.

Optimum production: - x_1= 100, that is, 100 bricks of type A are produced weekly and X_2 = 50, that is, 50 bricks of type B are produced weekly. The maximum profit is 1,050 million, ignoring the negative sign in the tableau.

Value of slack variables:

S_1 = 50, means that at optimum 50 machine hours are not used.

S_2 = 50, means that also at optimum 50 units of the special decorating material

are not used.

S_3 and **S4** have no unused capacity and thus fully utilised. They are said to **binding** and will always have **non-zero shadow prices.**

Additional explanation:

The values of Row 20 for the slack variables are of great importance. These are the valuations of resources and are known as ***shadow prices.*** These have the following meaning:

S_1 = 0, that is, there is no value gain from increasing machine hours.

S_2 = 0, that is, there is no value gain from increasing units of the special decorating material.

S_3 = -4, that is, for every extra kg of the setting chemical used 4 million extraoverall profit would be realised.

S_4 = -5, that is, for every extra brick of type B, the overall profit would increase by 5 million.

b) Minimization problem

Generally, a minimization problem can be solved by changing it to a maximisation problem since minimizing **Z** is the same as maximizing **- Z**. Also, constraints th5-involve the ≥ are converted to those with the sign ≤ . The conversion of both the objective function and the constraints affected is by multiplying through by -1.

Example 26.**6**

Given the following linear programming problem:

Minimise Z = $\mathbf{X_1 - 3x_2 + 2x_3}$

Subject to the constraints:

$3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x2 + 8x3 \leq 10$

$X_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

Required:

Solve the above LP problem by the simplex method.

Solution:

Note: In this minimisation problem, all the constraints are in the standard form of a maximisation problem. Therefore, only the objective function is converted.

Converting Z and introducing the slack variables:

Maximise$\mathbf{Z_1 = - X_1 + 3x_2 -} 2x_3$, that is, $\mathbf{Z_1}$= Z

$\mathbf{3x_1} - x_2 + \mathbf{2x_3} + s_1 = 7$

$-2x_1 + 4x_2 + s_2 = 12$

$-4X_1 + 3X_2 \mathbf{+} 8x_3 + \mathbf{S_3} = 10$

$\mathbf{X_1 \geq 0},\ \mathbf{x_2} \geq 0,\ x_3 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0$

2nd Tableau:

		Variable					Slack variable	
Row	**Solution Variable**	**Xi**	$\mathbf{x_2}$	$\mathbf{x_3}$	**Si**	$\mathbf{s_2}$	$\mathbf{S_3}$	**Solution**
1	**Si**	**3**	-1	2	1	0	0	7
2	$\mathbf{s_2}$	-2	4	0	0	1	0	12
3	**S3**	-4	**3**	8	0	0	1	10
4	**Z1**	-1	**3**	-2	0	0	0	0

Since greatest non-zero value of **Z1** is 3 (under **X2),** the pivot column and 12-h **4 -**

3< 10 ÷3, then $\mathbf{S_2}$is the pivot row. Thus 4 is pivot element

2nd tableau:

		Variable			Slack variable			
Row	Solution Variable	**Xi**	$\mathbf{x_2}$	$\mathbf{X_3}$	**Si**	$\mathbf{s_2}$	**S3**	Solution
R5(R1+R6)	**Si**	**5/2**	**0**	**2**	**1**	**1/4**	**0**	**10**
R6(R2/4)	$\mathbf{x_2}$	1/2	**1**	**0**	**0**	**1/4**	**0**	**3**
R7(R3-3R6)	**S3**	1/5	**0**	**8**	**0**	**3/4**	**1**	**1**
R8(R4-3R6)	**Zi**	1/2	**0**	**-2**	**0**	**b/4**	**0**	**-9**

Now greatest non-zero value Zi is 1/2(under Xi), the pivot column and $10{:}5/2 = 4$, < the only ratio >0, then $s1$ is the pivot row. Thus, - is pivot element

3rd Tableau:

		Varia		Die	slack Variable			
Row	Solution Variable	Xi	x_2	x_3	s ,	s_2	S_3	Solution
R9(R_5 5/2)	Xi	5/2	0	4/5	2/ 5	1/10	0	4
R10(R6+R9/2]	X2	0	1	2/5	1/5	3/10	0	5
R11(R7+fR9)	S_3	0	0	10	1	1 2	1	11
R12(R8-R9/2)	Z1	0	0	12/5	-1/5	-4/5	0	-11

Then $max.Zi$ **= 11, When** $xi = 4$, X2 **= 5,** x_3 **= 0,** Si= **0,** S2 **= 0,** S3 **= 11**

Mini.Z = -11, **Xi =** 4, X2= 5, X3= 0, Si= 0, S2 $= 0$, S3 $= 11$

LP Problems with mixed constraints

The LP problems with mixed constraints can be solved as illustrated in examples below:

Example 7

Given the LP problem below: Maximise Z = 20 xi + $15x2$ Subject to:

X, + X2 $\geq\geq$ 7

$9x_1 + 5x_2 \leq\leq 45$

$2x1 + x_2 \geq\geq 8$

X1 $\geq\geq 0$, $x_2 \geq\geq 0$

Required:

Solve the LP **problem using the simplex method.**

Solution:

Note: The problem is a maximisation but not in standard form.

1st conversion of the constraints

Max. Z = **20 x, + 15**x_2

Subject to:

-Xi - X2+S1 $= -7 = -7$

$9x_1 + 5x_2$ **= 45**

-2 x1 - $x_2 \leq -\leq -8$

X1 $\geq\geq$ **0** , $x_2 \geq\geq$ **0** ,

2nd introduction of the slack variables:

Max. Z = **20** xi + 15x2

Subject to:

-x1 - x2 +s1 = -7

9 **x1 + 5x2 +s2 = 45**

-2 x1 - x2 +s3 = -8

x1 > 0, x2 > 0, s1> 0, s2> 0, s3> 0

3rd Formulation of the initial tableau

	Variable		Slack variable			
Solution variable	X1	x_2	S1	s_2	s_3	Solution
Si	**-1**	**-1**	**1**	**0**	**0**	-7
s_2	9	**5**	**0**	**1**	**0**	45
S3	**-2**	**-1**	**0**	**0**	**1**	-8
Z	20	15		o	0	0

Observe that the solution represented in the initial tableau is not a feasible solution because of the two negative values (i.e., -7 and -8). In this case the solution is obtained in two phases.
4th determining when phase I or phase II applies:

Phase I applies when the table contains negative solutions while phase II is used when the non-negative condition in the standard maximisation problem is satisfied in the constraints.

5th choice of the pivot element:

For phase I, the pivot row is the one with the negative solution with the greatest magnitude. The pivot column is determined by obtaining the ratio with the negative entry under the variable and along the pivot as the numerator and the negative solution as the denominator. The bigger ratio identifies the pivot column.

However, should all entries along the pivot row under the variable columns be positive then the problem has no solution.

Phase II, is the approach used above when handling a standard maximization problem.

From the initial tableau, the solution column has two negative values. Therefore,

Phase I approach is applied first and since — > —| then -2 is the pivot element as identified below.

Initial tableau:

		Variable		**Slack variable**			
Row	**Solution variable**	**X,**	**x_2**	**Si**	**S_2**	**s_3**	**Solution**
R1	**s.**	-1	-1	1	0	0	-7
R2	**s_2**	9	5	0	1	0	**45**
R3	**s_3**	-2	-1	0	0	1	-8
R4	**Z**	**20**	**15**	**0**	**0**	**0**	**0**

2ndtableau:

		Variable		**Slack variable**			
Row	**Solution variable**	**Xi**	**x_2**	**S!**	**S_2**	**S_3**	**Solution**
R5(R1+R7)	**S**	0	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	**-3**
R6(R2/9-R7)	**S_2**	0	$\frac{1}{2}$	0	**1**	$\frac{9}{2}$	**9**
R7(R3/-2)	**X,**	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	**4**
R8(R4-20R7)	**Z**	**0**	**5**	**0**	**0**	**10**	**-80**

Because of -3 in the solution column then $\frac{11}{22}$ in R5 becomes next pivot element.

3[rd]tableau:

Row	Solution variable	X_1	X_2	S,	S_2	S_3	Solution
		Variable		Slack variable			
R9(-2R5)	x_2	0	**1**	-2	0	**1**	**6**
R10(R6-R9/2)	s_2	0	o	1	1	4	**6**
R11(R7-R9/2)	x,	1	**0**	1	0	-1	**1**
R12(R8-5R9)	Z	**0**	**o**	**10**	**0**	**s**	**-110**

At this point the non-negative requirement in the constraints is satisfied as shown in the 3[ra] tableau. However, the solution is improved further using the Phase II approach as in Example 1 above.

Since the highest non-negative entry under the variables along the Z row is 10, S_1 column becomes the pivot column and $\frac{11}{11}<\frac{99}{11}$, R11 is the pivot row.

4,[th] tableau:

		Variable		S	slack variable		
Row	**Solution variable**	Xi	x_2	Si	s_2	s_3	**Solution**
R13(R9+2R15)	x_2	2	**1**	0	0	**-1**	**8**
R14(R10-R15)	s_2	-1	**0**	0	I	5	**5**
R15(R11)	**Si**	1	**0**	1	0	-1	**1**
R16(R12-10R15)	Z	**-10**	**0**	**0**	**0**	**15**	**-120**

Because of 15 in the Z row, S3 is pivot column and the only positive ratio being $\frac{55}{55}$
R14 becomes the pivot row.

5[th] tableau:

		Variable		Slack variable			
Row	**Solution variable**	X_1	x_2	**Si**	s_2	$\mathbf{S_3}$	**Solution**
R17(R13+R18)	x_2	$\frac{9}{5}$	**1**	0	$\frac{1}{5}$	0	**9**
R18(R14/5)	$\mathbf{S_3}$	$-\frac{1}{5}$	**0**	0	$\frac{1}{5}$	1	**1**
R19(R15+R18)	$\mathbf{S_1}$	$\frac{4}{5}$	**0**	1	$\frac{1}{5}$	0	**2**
R20(R16-15R18)	Z	-7	**0**	**0**	-3	**0**	**-135**

The 5[th] tableau gives the optimum solution there being no positive values along Z column.
The optimum solution is Max. Z = 135
xi = 0, X2 = 9, si= 2, s_2= 0, s_3= 1

Example** 26.**8

An LP problem is given as;

Minimise Z = $5x_1 + 6x_2$

Subject to:

$X_1 + x_2 \leq 10$

$X_1 + 2x_2 \geq 12$

$2x_1 + x_2 \geq 12$

$X_1 \geq 3$

$X_1 \geq 0$, $X_2 \geq 0$

Required:

Solve the given LP problem using the simplex method.

Solution:

Step 1: Changing from the minimisation problem to the maximisation problem, that is:

Min. Z = Max. Zi(-Z) = $-5x1 - 6x_2$

Subject to:

$X_1 + x_2 \leq 10$

$-x_1 - x_2 \leq -12$

$-2x_1 - x_2 \leq -12$

$-X_1 < -3$

$X_1 \geq 0$, $x_2 \geq 0$

Step 2: Introduction of the slack variables to the constraints:

Max. $Z_1 = -5x_1 - 6x_2$

Subject to:

x1 + x2 + s1 = 10

-2x1 -x2 + s2 = -12

-2x1 -x2 + s3 = -12

-x1 + s4 = -3

x1≥0, x2 ≥ 0, s1 ≥0, s2 ≥ 0, S3≥ 0, s4 ≥ 0

Step* 3:** Setting up the initial tableau and proceed to improve the solution. ***Initial tableau:

		Variable		*Slack variable*				
Row	***Solution variable***	X_1	x_2	s_1	s_2	s_3	s_4	***Solution***
R1	***Si***	*1*	*1*	*1*	*0*	*0*	*0*	*10*
R2	S_2	*-1*	*-2*	*0*	*1*	*0*	*0*	*-12*
R3	S3	*-2*	*-1*	*0*	*0*	*1*	*0*	*-12*
R4	s_4	*-1*	*0*	*0*	*0*	*0*	*1*	*-3*
R5	***Zi***	*-5*	*-6*	*0*	*0*	o	*0*	*0*

Implementing

Phase I: Because of ***-12*** and ***-3*** in the solution column, ***R2*** or ***R3*** can be the pivot row. In this case ***R2*** has been used.

Row	*Solution variable*	*X,*	x_2	**s,**	s_2	S_3	s_4	*Solution*
R6(R1-R7)	*Si*	**1/2**	*0*	*1*	**1/2**	*0*	*0*	*4*
R7(R2/-2)	x_2	**1/2**	*1*	*0*	1	*0*	*0*	**6**
R8(R3+R7)	**S3**	*-3/2*	*0*	*0*	**-1/2**	*1*	*0*	**-6**
R9(R4)	S_4	*-1*	*0*	*0*	*0*	*0*	*1*	**-3**
R10(R5-6R7)	*z,*	*-2*	*0*	*0*	**-3**	*0*	*0*	**36**

Because of *-6* and *-3* in the solution column Phase I continues; ***R3*** is pivot row and ***xi*** is pivot column.

3rd tableau

		Variable		*Slack variable*				
Row	*Solution variable*	*X,*	x_2	*Si*	s_2	S_3	s_4	*Solution*
R11(R6+R13/2)	*S1*	*0*	*0*	*1*	*1/3*	*1/3*	*0*	**2**
R12(R7+R13/2)	x_2	*0*	*1*	*0*	*-2/8*	*1/3*	*0*	*4*
R13(-2R8/3)	*X1*	*1*	*0*	*0*	*1/3*	**-2/3**	*0*	*4*
R14R9+R13)	S_4	*0*	*0*	*0*	*1/3*	**-2/3**	*1*	*1*
R15R10+213)	*z,*	**0**	*0*	*0*	**-7/3**	*-4/3*	*0*	*44*

Phase I ends and there being positive values along the Z row under the variable entries, the optimum solution has therefore been achieved.

The optimum solution is;

Maximum $\mathbf{Z_1}$=-44, hence, minimum **Z** = 44.

When x_1 = 4, $\mathbf{X_2}$= 4, si = 2, $\mathbf{S_2}$= 0, $\mathbf{S_3}$= 0, $\mathbf{S_4}$= 1.

An alternate approach for solving a minimisation LP problem in which all the constraints contain the ≥ sign:

***Example** 26.9*

A fruit packing factory uses two products, the preservative and for flavour in varying quantities to produce three brands A, B and C. The factory wishes to produce at least 150 units of A, 200 units of B, and 60 units of C. Each ton of the preservative yields 3 of A, 5 of B, and 3 of C. Each ton of the flavour product yields 5 of A. 5 of Band 1 of C.

The cost of the preservative is 40 million per ton and that of the flavour is 50 million per ton.

Required:

Advise the factory manager on how to minimise the cost of production.

Solution:

Let x_1 = the number of tons of the preservative used,

$\mathbf{X_2}$= the number of tons of the flavour used.

The original or the given minimisation problem also known as the primal**:**

Minimise, $40x_1 + 50x_2$

Subject to:

$3x_1 + 5x_2 \geq 150$ (brand A)

$5x_1 + 5x_2 \geq 200$ (brand B)

$3x_1 + x_2 \geq 60$ (brand C)

$X_1 \geq 0$, $x \geq 0$

The maximisation problem is then obtained from the primal, also referred *to* as the **inverse** or the **dual.**

This is achieved by operating along the columns of the primal as illustrated below;

Maximise 150A + 200B + 60C

Subject to:

3A + 5B + 3C ≤ 40

5A + 5B + C ≤50

It is important to note that the original quantity column (i.e., 150, 200 and 60) has become the objective function and the original costs (i.e., 40 and 60) have become the amounts of the constraints.

After the transformation, the constraints are then converted to equations. The initial tableau is then set up as below and the rest of the process of improvement is implemented accordingly:

Maximise 150 A + 200B + 60C

Subject to:

3A + 5B + 3C + Si = 40

5A + 5E + C + s_2 = 50

Initial tableau:

Solution variable	A	B	C	S_1 (Slack variables)	S_2 (Slack variables)	Cost
S_1	3	[5]	3	**1**	0	40
S_2	5	5	1	0	1	50
Quantity	150	200	60	0	0	0

Then pivot element is under B, and from 40 ÷ 5 =8, 50÷5 = 10, the pivot is as indicated the initial tableau.

2[nd]**table**

Row	Solution variable	A	B	C	S_1 (Slack variables)	S_2 (Slack variables)	Cost
1(rows$_1$ ÷5)	**B**	$\frac{11}{55}$	1	$\frac{11}{55}$	$\frac{11}{55}$	0	8
2(as rows$_2$)	**S_2**	5	5	1	0	1	50
3(as quantity row)	**Quantity**	150	200	60	0	0	0

9. Optimum solution

The figures under the slack variable columns represent the quantities to be purchased. The -25 in the s1 column means, purchase 25 tons of the preservative and the -15 in the s2 column means purchase 15 tons of the flavour.

This yield a minimum cost of 1,750 million (coming from the -1,750 under the cost column).

Note: 25 x 40m+ 15 x 50m = 1,750m.

10. Advantages of the simplex method

The simplex method has a number of advantages associated with it among which are the following:
It can deal with a problem having two or more variables.
The final tableau gives the optimum solution including the shadow prices or costs.
Each step in the procedure is summarised in a table form known as a tableau.

11. Disadvantages of the simplex method

The method is quite challenging and becomes rather complex with the increase in the number of variables.

Conversion from the primal to dual problem and therefore after interpretation of the final tableau pauses additional challenges.

12. Advantages of the dual method

It is a maximisation model whose solution is straight forward by simplex method.

It may simplify the solution method, for example, primal with three decision variables and two constraints giving rise to a dual that can be solved graphically.

13. Disadvantages of the dual method

It is tedious, especially when it involves many variables.

Self-test questions

Question 1

A factory is considering buying two machines M1 and M2 which make a special component.

Information about the machines is given in the table below.

Machine	Hourly output	Hourly profit	Floor space
M1	60	Frw 24,000	30m^2
M2	40	Frw 15,000	24m^2

The factory is to buy at least as many fvh machines as **Mi**machines. At least 480 components must be produced hourly and up to 360m^2 of floor space is available.

Required:

a) Write down the linear inequalities to this problem.

b) Indicate the feasible region on a graph paper.

c) Find the combination of machines that should be purchased in order to maximise profit.

d) Find how much floor space remains.

Solution:

a) x = No of M1 machine

y = No of M2 machines

Then ***y* > *x***

3x + ***2y*** **>**24

5x* + *4y <60

x*.,*y≥ 0

c) Objective function

Maximize P = 24,000 + 15000y

4 M1 and 10 M2 should be purchased

d) Space used = 320 m^2 remaining space = 40 m*

Question 2

A retailer deals in two items only item P and item Q. He has Frw 500,000 to invest and a space to store at most 60 pieces. An item P costs him Frw 25,000 and item Q costs him Frw5.000. A net profit to him on item P is Frw 5,000 while on item Q is Frw 1,500. If he can sell all the items that he purchases.

Required:

a) Give the mathematical formulation to the above problem.

b) Represent the linear inequalities on the graph paper.

c) If he can sell all the items that he purchases, how should he invest his amount to have amaximum profit?

Solution:

a) x = items of type P

y = items of type Q

Objective function:

Maximize profit

(Z) = 500x + 1,500y subject to:

5x + *y* ≤100

x + *y* ≤60

x ,*y* ≥ 0

c) To max Z, 10 P items should be produced and 50 Q items.

Amount to invest = Frw 500,000.

Question 3

A cold drinks company has two bottling plants, located at two different places. Each plant produces three different drinks A, B and C. The capacities of two plants, in number of bottles per day are as follows:

	Drink A	**Drink B**	**Drink C**
Plant 1	3000	1000	2000
Plant 11	1000	1000	6000

A market survey indicates that during any particular month there will be a demand of 24,000bottles of A, 16,000 bottles of B and 48,000 bottles of C. The operating costs per day, or running plants 1 and 11 are, respectively, 600 monetary units and 400 monetary units.

How many days should the company run each plant during the month so that the production cost is minimised while still meeting the market demand?

Solution:

a) ***3x + y*** ≥24

x + y ≥ 16

2x **+** 5y ≥ 40

x,y≥0

Where x and y are running days for plants 1 and 2. respectively in a month.
c) Running days to minimize the cost.
4 days of plant 1.
12 days of plant 2.

Question 4

Acertain company produces two types of dining tables $D_1$and D_2. The profits made on a tableare Frw 50,000 and Frw75,000, respectively. Type D_1requires 4 hours to assemble and $2^1/_4$ hours to finish. Type $\mathbf{D_2}$requires 2 hours of assembling and 3 hours of finishing. There are12 hours of assembling and 15 hours of finishing.

Required:

a) Use inequalities to represent this information.

b) Use graphical method to find the maximum possible number of dining tables of each type that can be produced.

c) Calculate the maximum profit obtained.

Solution:

a) *2x + y* ≤6
5x + 6y ≤ 30
x ,y≥0
c) Max profit P = Frw350,000.

Question 5

A tours company has two types of buses, Scania and Isuzu. It has four Isuzu buses each of which can take a maximum of 50 passengers and three Scania buses, each of which can carry up to 60 passengers. The company has a total of five drivers available. The hire charges per kilometre are Frw 10,000 for an Isuzu bus and Frw 12,000 for a Scania bus. An association wishes to hire buses to a town 40km away and it has Frw 3 million for to and fro. By letting xand y to be the number of Isuzu and Scania buses the association hires, respectively.

Required:

a) Write down an expression for the members who go on the journey.

b) Write down inequalities in x and y to represent the:

i. number of each type of bus used;

ii. total number of drivers employed; and

iii. amount of money spent on the journey.

c) c) Represent the inequalities on the same graph.

d) d) Use the graph to determine the maximum number of members of the association who go on the journey.

Solution:

a) Number N people who go for the trip.

N = 50x + 60y

x = No. of ISUZU buses hired,

y = No. of SCANIA buses hired.

b) i) $0 \leq x \leq 4$, $0 \leq y \leq 3$

ii) $x + y \leq 5$

iv) $20X + 24y \geq 75$

d)To maximize N, the number of buses hired are:

x = 2 ISUZU buses

y = 3 SCANIA buses

Maximum value of N = 280

Question 6

a) Define the following terms:

i) Primal.

ii) Dual.

b) Briefly describe the steps involved when converting a primal to a dual.

Solution:

a) The answer to i) and ii) can be located in Chapter 26, Section 9c).

b) The answer can be located in Chapter 26 on simplex method under b) minimisation.

Question 7

A firm produces three products X, Y and Z each with a contribution of £20, £18 and £16, respectively. Production data are as follows:

	per unit		
products	**machines hours**	**Labour Hours**	**materials(kg)**
X	5	2	8
Y	3	5	10
Z	6	3	3
Availability	3000 **hours**	2,500 **hours**	10,000 **kg**

Set up the initial simplex tableau.

Solution:

Solution variable	**X**	**Y**	**Z**	**S,**	S_2	s_3	**Solution**
S_1	5	3	6	1	0	0	3,000
S_2	2	5	3	0	1	0	2,500
S_3	8	10	3	0	0	1	10,000

	20	18	16	0	0	0	0

Question 8

Improve the initial tableau in question 7, above once.

Solution:

Solution variable	**X**	Y	**Z**	**Si**	**s_2**	**S_3**	**Solution**
		3	6	1			
X	1	$\frac{3}{5}$	$\frac{66}{55}$	$\frac{11}{55}$	0	0	600
	1	$3\frac{44}{55}$	3 5	$-\frac{2}{5}$	1	0	1,300
S3	0	5 $\frac{11}{55}$	$-6\frac{33}{55}$	$-1\frac{2}{5}$	0	1	5,200
	0	6	-8	-4	0	0	

Question 9

Interpret the final tableau of a simplex solution shown below:

	X	**Y**	**z**	**S1**	**S2**	**S3**	**S4**	
s_2	0	**0**	0	**1.25**	1	2	-0.5	300
Z	1	0	0	0.75	0	1.25	3	250
X	0	1	0	0	0	2	1	70
Y	0	0	1	-0.5	0	-1.5	2.5	122
	0	**0**	**0**	**-4**	0	-10.2	-12	45,000

Where x = units of product X

y = units of product Y

z = units of product Z

S_1= slack variable for labour hours

S_2= slack variable for material (kg)

S_3= slack variable for machine hours

S_4 = slack variable for sales restriction on X

Solution:

For optimum productions:

Produce:

70 units of product X, 122 units of product Y, 250 units of product Y

And this gives a contribution of £45,000.

In addition, there are 300kg of unused material and the following shadow prices:

Shadow price **Si** = £4 per labour hour, Shadow price **s_3**= £10.2 per machine hour

Shadow price s_4 = £12 per unit

Question 10

Determine optimal values of **Xi** and $\mathbf{x_2}$ for linear programming problem below using the simplex method

Maximise, $\mathbf{8x_1 + 10x_2}$

Subject to:

$X_1 + x_2 \leq 80$

$2x_1 + x_2 \leq 100$

$X_1 + 2X_2 \leq 120$

$X_1, X_2 \geq 0$

Solution:

xi =27 and x_2= 47, giving a maximum contribution of 8 x 27 + 10 x 47 = 686, the values of **x1** and x_2 given to the nearest unit

Question 11

Use the simplex method to maximise $Z = 4x_1 + 8x_2 + 10x_3$ Subject to:

$X_1 + x_2 + x_3 \leq 1000$

$0.5x_1 + 0.5X_2 + 0.5X_3 \leq 1500$

$0.5xi + X2 + x_3 \leq 2000$

$X_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$

In question 12 and 13 form the dual of the given primal problem.

Solution:

$X_1 = 0$, $x_2 = 0$, $x_3 = 1000$ a

Question 12

Minimise $30xi + 60x_2 + 2i$

Subject to:

$5x_1 + 10x_2 + 15X_3 \geq 2000$

$2x_1 + 3x_2 + X_3 \geq 300$

$3x_1 + 6x2 + 4X_3 \geq 650$

$x_1 \geq 0$, $x_2 \geq 0$, $X_3 \geq 0$.

Solution*:*

Maximise, $2000P_1 + 300P_2 + 650P_3$

Subject to:

$5P_1 + 2P_2 + 8P_3 \leq 30$

$10P_1 + 3P_2 + 6P_3 \leq 60$

$15P_1 + P_2 + 4P_3 \leq 20$

$P_1 \geq 0$, $P_2 \geq 0$ and $P_3 \geq 0$

Question 13

Minimise $200x_1 + 250x_2 - 100xs + 50x_4$

Subject to:

$6x_1 + 10x_2 - 3X_3 \geq 55$

$9x_1 + 7x_2 + 3X_4 \geq 40$

$X_1 \geq 0,\ x_2 \geq 0,\ x_3,\ x_4 \geq 0$

Solution:

Maximise, $55P_1 + 40P_2$

Subject to:

$6P_1 + 9P_2 \leq 200$

$10P_1 + 7P_2 \leq 250$

$-3P_1 \leq 100$

$3P_2 \leq 50$

Pi ≥ 0 and $P_2 \geq 0$

Question 14

Minimise, $Z = 6x_1 + 8x_2 + X_3$

Subject to:

$3x_1 + 5x_2 + 3X_3 \geq 20$

$xi + 3x_2 + 2X_3 \geq 9$

$6x_1 + 2x_2 + 5X_3 \geq 30$

$X_1 + X_2 + X_3 \leq 10$

$X_1 \geq 0,\ x_2 \geq 0,\ X_3 \geq 0$

26.7. Transportation problem

26.7.1. Aims and objectives

In this unit we would be able to learn the Time Management Models. i.e. Transportation and Assignment Models, thus would be able to learn transportation problems deal with the transportation of a product manufactured at different plants *(supply origins)* to a number of different warehouses *(demand destinations).* The objective is to satisfy the demand at destinations from the supply constraints at the minimum transportation cost possible. To achieve this objective, we must know the quantity of available supplies and the quantities demanded. In addition, we must also know the location, to find the cost of transporting one unit of commodity from the place of origin to the destination. The model is useful for making strategic decisions involved in selecting optimum transportation routes so as to allocate the production of various plants to several warehouses or distribution centers.

The transportation model can also be used in making location decisions. The model helps in locating a new facility, a manufacturing plant or an office when two or more number of locations is under consideration. The total transportation cost, distribution cost or shipping cost and production costs are to be minimized by applying the model.

26.7.2. Introduction

Transportation problem is a particular class of linear programming, which is associated with day-to-day activities in our real life and mainly deals with logistics. It helps in solving problems on distribution and transportation of resources from one place to another. The goods are transported from a set of sources (e.g., factory) to a set of destinations (e.g., warehouse) to meet the specific requirements. In other words, transportation problems deal with the transportation of a product manufactured at different plants *(supply origins)* to a number of different warehouses *(demand destinations).* The objective is to satisfy the demand at destinations from the supply constraints at the minimum transportation cost possible. To achieve this objective, we must know the quantity of available supplies and the quantities demanded. In addition, we must also know the location, to find the cost of transporting one unit of commodity from the place of origin to the destination. The model is useful for making strategic decisions involved in selecting optimum transportation routes so as to allocate the production of various plants to several warehouses or distribution centers.

The transportation model can also be used in making location decisions. The model helps in locating a new facility, a manufacturing plant or an office when two or more number of locations is under consideration. The total transportation cost, distribution cost or shipping cost and production costs are to be minimized by applying the model

26.7.3. Mathematical formulation

The transportation problem applies to situations where a single commodity is to be transported from various sources of supply **(origins)** to various demands **(destinations).**

Let there be *m* sources of supply $S_1, S_2, \ldots\ldots S_m$ having a_i (i = 1,2,......m) units of supplies respectively to be transported among *n* destinations $D_1, D_2 \ldots\ldots D_n$ with b_j (j = 1,2…..n) units of requirements respectively.

Let C_{ij} be the cost for shipping one unit of the commodity from source i, to destination j for each route. If x_{ij} represents the units shipped per route from source i, to destination j, then the problem is to determine the transportation schedule which minimizes the total transportation cost of satisfying supply and demand conditions.

The transportation problem can be stated mathematically as a linear programming problem as below:

$$Minimize\ Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} = 1$$

Subject to constraints,

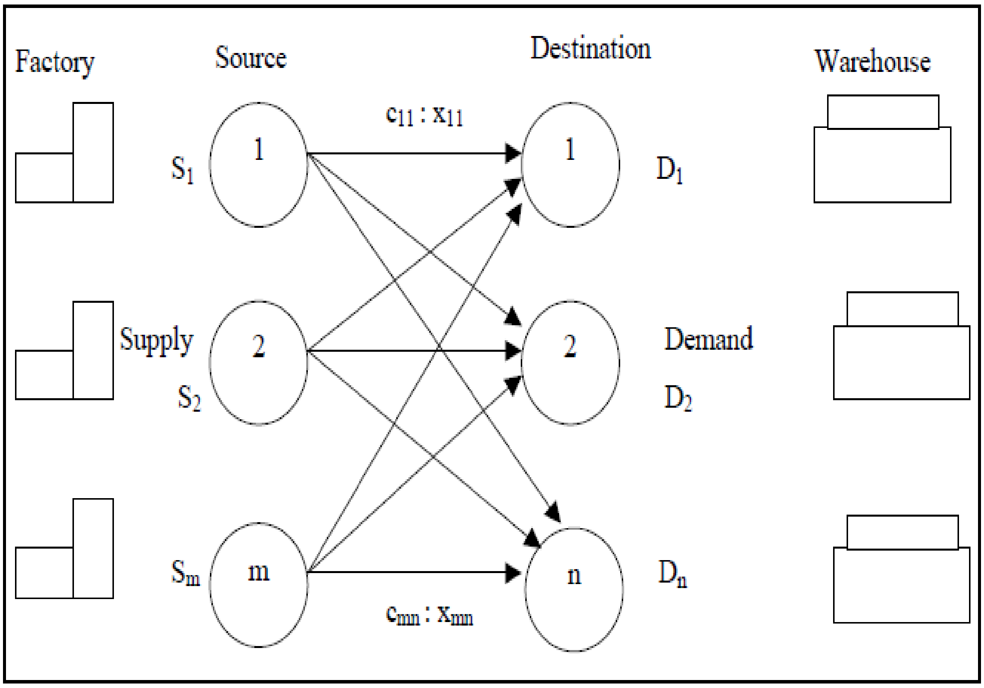
$$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1,2, \ldots., m \ (supply\ constraints)$$

$$\sum_{i=1}^{n} x_{ij} = b_j \quad j = 1,2, \ldots., m \ (demand\ constraints)$$

And $x_{ij} \geq 0$ $x_{ij} \geq 0$ for all $i = 1,2, \ldots., m\ and\ j = 1,2, \ldots., m$ $i = 1,2, \ldots., m\ and\ j = 1,2, \ldots., m$

26.7.4. **Network Representation Of Transportation** Model

The transportation model is represented by a network diagram below



where,

m be the number of sources,

n be the number of destinations,

S_m be the supply at source m,

D_n be the demand at destination n,

To / From	D_1	D_2	...	D_n	Supply A_i
S_1	C_{11} x_{11}	C_{12} x_{12}	...	C_{1n}	A_1
S_2	C_{21} x_{21}	C_{22} x_{22}	...	C_{2n}	A_2
. . .	. . .	. . .	... ...	. . .	. . .
S_m	C_{m1} x_{m1}	C_{m2} x_{m2}	...	C_{mn}	A_m
B_j	B_1	B_2	...	B_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

c_{ij} be the cost of transportation from source i to destination j, and
x_{ij} be the number of units to be shipped from source i to destination j.

The objective is to minimize the total transportation cost by determining the unknowns x_{ij}, i.e., the number of units to be shipped from the sources and the destinations while satisfying all the supply and demand requirements.

26.7.5. General representation of transportation model

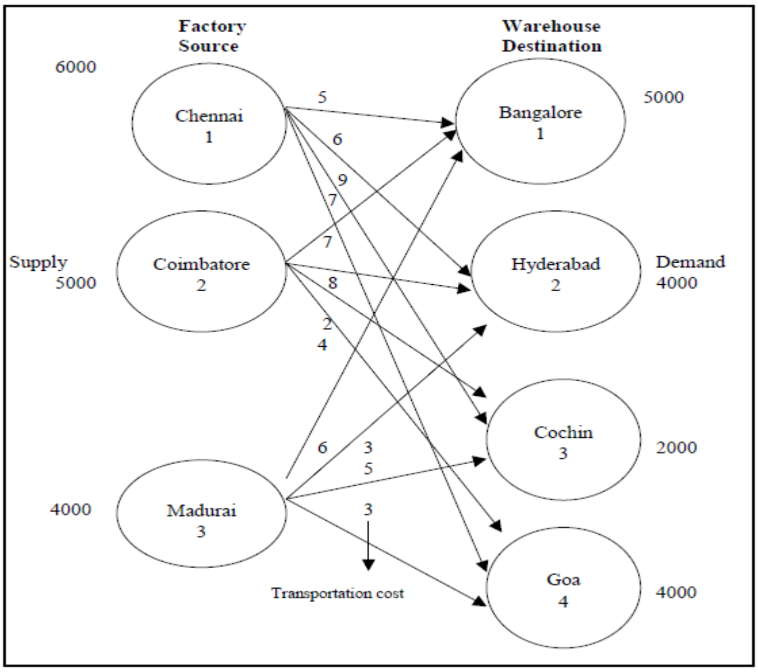
The Transportation problem can also be represented in a tabular form as shown in above

Let C_{ij} be the cost of transporting a unit of the product from i^{th} origin to j^{th} destination. a_i be the quantity of

7. the commodity available at source i,

b_j be the quantity of the commodity needed at destination j, and x_{ij} be the quantity transported from i^{th} source to j^{th} destination

26.7.6. Use of linear programming to solve transportation problem



26.7.7. Formulation of LP model
Example 26.7.1

The management of UTEXITRWA would like to determine the number of units to be shipped from each textile unit to satisfy the demand of each wholesale distributor. The supply, demand and transportation cost are as follows:

Production Capacities

Supply	Textile Unit	Weekly Production (Units)
1	Chennai	6000
2	Coimbatore	5000
3	Madurai	4000

Demand Requirements

Destination	Wholesale Distributor	Weekly Demand (Units)
1	Bangalore	5000
2	Hyderabad	4000
3	Cochin	2000
4	Goa	4000

Transportation cost per unit

Supply	Destination			
	B'lore	Hyderabad	Cochin	Goa
Chennai	5	6	9	7
Coimbatore	7	8	2	4
Madurai	6	3	5	3

Required:

d) Formulate linear LP model

Objective function: The objective is to minimize the total transportation cost. Using thecost data table, the following equation can be arrived at:

Transportation cost for units
shipped from Chennai = $5x_{11}+6x_{12}+9x_{13}+7x_{14}$

Transportation cost for units

shipped from Coimbatore = $7x_{21}+8x_{22}+2x_{23}+4x_{24}$ Transportation cost for units

shipped from Madurai= $6x_{31}+3x_{32}+5x_{33}+3x_{34}$

Combining the transportation cost for all the units shipped from each supply point with the objective to minimize the transportation cost, the objective function will be,

Minimize Z = $5x_{11}+6x_{12}+9x_{13}+7x_{14}+7x_{21}+8x_{22}+2x_{23}+4x_{24}+6x_{31}+3x_{32}+5x_{33}+3x_{34}$

Constraints:

In transportation problems, there are supply constraints for each source, and demand constraints for each destination.

Supply constraints:
For Chennai, $x_{11}+ x_{12}+ x_{13}+ x_{14} \leq 6000$
For Coimbatore, $x_{21}+ x_{22}+ x_{23}+ x_{24} \leq 5000$
For Madurai, $x_{31}+ x_{32}+ x_{33}+ x_{34} \leq 4000$
Demand constraints:

For B'lore, $x_{11} + x_{21} + x_{31} = 5000$
For Hyderabad, $x_{12} + x_{22} + x_{32} = 4000$
For Cochin, $x_{13} + x_{23} + x_{33} = 2000$
For Goa, $x_{14} + x_{24} + x_{34} = 4000$
Simply

The linear programming model for GM Textiles will be write in the next line. Minimize $Z = 5x_{11} + 6x_{12} + 9x_{13} + 7x_{14} + 7x_{21} + 8x_{22} + 2x_{23} + 4x_{24} + 6x_{31} + 3x_{32} + 5x_{33} + 3x_{34}$

Subject to constraints,

$X_{11}+x_{12}+x_{13}+x_{14} \leq 6000$ ....(i)

$x_{21}+x_{22}+x_{23}+x_{24} \leq 5000$ ....(ii)

$x_{31}+x_{32}+x_{33}+x_{34} \leq 4000$ ....(iii)

$x_{11}+x_{21}+x_{31} = 5000$ ....(iv)

$x_{12}+x_{22}+x_{32} = 4000$ ....(v)

$x_{13}+x_{23}+x_{33} = 2000$ ....(vi)

$x_{14}+x_{24}+x_{34} = 4000$ ....(vii)

Example 26.7**:** Consider the following transportation problem (Table 6.5) and develop a linear programming (LP) model.

Table 6.5: Transportation Problem

Source	**Destination**			
	1	**2**	**3**	**Supply**
1	15	20	30	**350**
2	10	9	15	**200**
3	14	12	18	**400**
Demand	**250**	**400**	**300**	

Solution: Let x_{ij} be the number of units to be transported from the source i to the destinationj, where i = 1, 2, 3,…m and j = 1, 2, 3,…n.

The linear programming model is

Minimize $Z = 15x_{11}+20x_{12}+30x_{13}+10x_{21}+9x_{22}+15x_{23}+14x_{31}+12x_{32}+18x_{33}$ Subject to constraints,

$x_{11}+x_{12}+x_{13} \leq 350$..................(i)

$x_{21}+x_{22}+x_{23} \leq 200$..................(ii)

$x_{31}+x_{32}+x_{33} \leq 400$..................(iii)

$x_{11}+x_{12}+x_{31} = 250$..................(iv)

$x_{12}+x_{22}+x_{32} = 400$.................. (v)

$x_{13} + x_{23} + x_{33} = 300$ .... (vi)

$x_{ij} \geq 0$ for all i and j.

In the above LP problem, there are m × n = 3 × 3 = 9 decision variables and m + n = 3 + 3 = 6 constraints.

26.7.8. Balanced transportation problem

When the total supplies of all the sources are equal to the total demand of all destinations, the problem is a ***balanced transportation problem.***

Total supply = Total demand

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

26.7.9. Unbalanced transportation problem

When the total supply of all the sources is not equal to the total demand of all destinations, the problem is an ***unbalanced transportation problem.***

$$Total\ supply \neq Total\ demand$$

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

Demand Less than Supply

In real-life, supply and demand requirements will rarely be equal. This is because of variation in production from the supplier end, and variations in forecast from the customer

These unbalanced problems can be easily solved by introducing ***dummy sources*** and ***dummy destinations.*** If the total supply is greater than the total demand, a dummy destination **(dummy column)** with demand equal to the supply surplus is added. If the total demand is greater than the total supply, a dummy source (dummy row) with supply equal to the demand surplus is added. The unit transportation cost for the dummy column and dummy row are assigned zero values, because no shipment is actually made in case of a dummy source and dummy destination.

Example 2: Check whether the given transportation problem shown in Table 6.6 is a balanced one. If not, convert the unbalanced problem into a balanced transportation problem.

Table 6.6: Transportation Model with Supply Exceeding Demand

Source	**Destination**			**Supply**
	1	**2**	**3**	
1	25	45	10	200
2	30	65	15	100
3	15	40	55	400
Demand	200	100	300	

Solution: For the given problem, the total supply is not equal to the total demand.

Since $\sum_{i=1}^{3} a_i = 700 \sum_{i=1}^{3} a_i = 700$ and $\sum_{j=1}^{3} b_j = 600 \sum_{j=1}^{3} b_j = 600$

The given problem is an unbalanced transportation problem. To convert the unbalanced transportation problem into a balanced problem, add a dummy destination (dummy column). i.e., the demand of the dummy destination is equal to,

$$\sum_{i=1}^{3} a_i \neq \sum_{j=1}^{3} b_j$$

Thus, a dummy destination is added to the table, with a demand of 100 units. The modified table is shown in below which has been converted into a balanced transportation table. The unit costs of transportation of dummy destinations are assigned as zero.

Table 6.7: Dummy Destination Added

Source	**Destination**				**Supply**
	1	**2**	**3**	**4**	
1	25	45	10	0	200
2	30	65	15	0	100
3	15	40	55	0	400
Demand	200	100	300	100	**700/700**

Example 3: Convert the transportation problem shown in Table 6.8 into a balanced problem.

Demand Exceeding Supply

Source	**Destination**				**Supply**
	1	**2**	**3**	**4**	
1	10	16	9	12	200
2	12	12	13	5	300
3	14	8	13	4	300
Demand	100	200	450	250	**1000/800**

Solution: The given problem is,

$\sum_{i=1}^{3} a_i = 800 \sum_{i=1}^{3} a_i = 800$ and $\sum_{j=1}^{3} b_j = 1000 \sum_{j=1}^{3} b_j = 1000$

The given problem is an unbalanced one. To convert it into a balanced transportation problem, include a dummy source (dummy row) as shown in Table 6.9

Table 6.9: Balanced TP Model

26.7.10. Procedure to solve transportation problem

Step 1: Formulate the problem.

Formulate the given problem and set up in a matrix form. Check whether the problem is a balanced or unbalanced transportation problem. If unbalanced, add dummy source (row) or dummy destination (column) as required.

Step 2: Obtain the initial feasible solution.

The initial feasible solution can be obtained by any of the following three methods:

Northwest Corner Method (NWC)

Least Cost Method (LCM)

Vogel's Approximation Method (VAM)

The transportation cost of the initial basic feasible solution through Vogel's approximation method, VAM will be the least when compared to the other two methods which gives the value nearer to the optimal solution or optimal solution itself. Algorithms for all the three methods to find the initial basic feasible solution are given.

Algorithm for North-West Corner Method (NWC)

e) Select the North-west (i.e., upper left) corner cell of the table and allocate the maximum possible units between the supply and demand requirements. During allocation, the transportation cost is completely discarded (not taken into consideration).

ii) Delete that row or column which has no values (fully exhausted) for supply or demand.

iii) Now, with the new reduced table, again select the North-west corner cell and allocate the available values.

iv) Repeat steps (ii) and (iii) until all the supply and demand values are zero.

v) Obtain the initial basic feasible solution.

Algorithm for Least Cost Method (LCM)

i) Select the smallest transportation cost cell available in the entire table and allocate the supply and demand.

ii) Delete that row/column which has exhausted. The deleted row/column must not be considered for further allocation.

iii) Again select the smallest cost cell in the existing table and allocate. (Note: In case, if there are more than one smallest costs, select the cells where maximum allocation can be made)

iv) Obtain the initial basic feasible solution.

Algorithm for Vogel's Approximation Method (VAM)

i) Calculate penalties for each row and column by taking the difference between the smallest cost and next highest cost available in that row/column. If there are two smallest costs, then the penalty is zero.

ii) Select the row/column, which has the largest penalty and make allocation in the cell having the least cost in the selected row/column. If two or more equal penalties exist, select one where a row/column contains minimum unit cost. If there is again a tie, select one where maximum allocation can be made.

iii) Delete the row/column, which has satisfied the supply and demand.

IV) Repeat steps (i) and (ii) until the entire supply and demands are satisfied.

v) Obtain the initial basic feasible solution.

Remarks: The initial solution obtained by any of the three methods must satisfy the following conditions:

The solution must be feasible, i.e., the supply and demand constraints must be satisfied (also known as rim conditions).

The number of positive allocations, N must be equal to m+n-1, where m is the number of rows and n is the number of columns.

Example 26.7.***4:*** The cost of transportation per unit from three sources and four destinations are given in Table 6.12. Obtain the initial basic feasible solutions using the following methods.

a) North-west corner method

b) Least cost method

c) Vogel's approximation method

Table 6.12: Transportation Model

Source	**Destination**				**Supply**
	1	**2**	**3**	**4**	
1	4	2	7	3	250
2	3	7	5	8	450
3	9	4	3	1	500
Demand	200	400	300	300	**1200**

Solution: The problem given in Table 6.13 is a balanced one as the total sum of supply is equal to the total sum of demand. The problem can be solved by all the three methods.

North-West Corner Method: In the given matrix, select the North-West corner cell. The North-West corner cell is (1,1) and the supply and demand values corresponding to cell (1,1) are 250 and 200 respectively. Allocate the maximum possible value to satisfy the demand from the supply. Here the demand and supply are 200 and 250 respectively. Hence allocate 200 to the cell (1,1) as shown in Table 6.13.

Table 6.13: Allocated 200 to the Cell (1, 1)

		Destination				**Supply**
		1	**2**	**3**	**4**	
Source	**1**	200 4	2	7	3	250 50
	2	3	7	5	8	450

3		9	4	3	1	5 0 0
Demand		200 0	400	3 0 0	300	

Now, delete the exhausted column 1 which gives a new reduced table as shown in Tables 6.14 (a, b, c, d). Again repeat the steps.

Table 6.14 (a): Exhausted Column 1 Deleted

		Destination			
		2	3	4	**Supply**
	1	2 5 0	7	3	50 0
Source	**2**	7	5	8	450
	3	4	3	1	500
Demand		400 350	300	350	

Table after deleting Row 1

Table 6.14 (b): Exhausted Row 1 Deleted

	Destination			
	2	**3**	**4**	**Supply**
2	7	5	8	450 100

Source 3	350 4	3	1	500
Demand	350 0	300	300	

Table after deleting column 2

Table 6.14 (c): Exhausted Column 2 Deleted

		Destination		
		3	**4**	**Supply**
Source	**2**	5 100	8	100 0
	3	3	1	500
Demand		**300** 200	300	

Finally, after deleting Row 2, we have

Table 6.14 (d): Exhausted Row 2 Deleted

		Destination		
		3	4	**Supply**
Source	**3**	3 300	1 200	**500**

Demand	300	200
	0	0

Now only source 3 is left. Allocating to destinations 3 and 4 satisfies the supply of 500.
The initial basic feasible solution using North-west corner method is shown in Table 6.15
Table 6.15: Initial Basic Feasible Solution Using NWC Method

	P ta					
	1	2	3	4	5	S lyp
1	0	2	3 (25)	5	9	25
2	5	0 (20)	15	2 (10)	4	30
Warehouse 3	5	5	4 (2)	7	5	20
4	2	5	3 (5)	¥	8 (25)	30
D e a	20	20	30	10	25	105

Transportation cost (4 × 200) + (2 × 50) + (7 × 350) + (5 × 100) + (2 × 300) + (1 × 300)

vi) 800 + 100 + 2450 + 500 + 600 + 300

vii) **Rs. 4,750.00**

Least Cost Method

Select the minimum cost cell from the entire Table 6.16, the least cell is (3,4). The corresponding supply and demand values are 500 and 300 respectively. Allocate the maximum possible units. The allocation is shown in Table 6.16.

Table 6.16: Allocation of Maximum Possible Units

Source	Destination 1	2	3	4	Supply
1	4	2	7	3	250
2	3	7	5	8	450
3	9	4	3	1 300	~~500~~ 200
Demand	200	400	300	~~300~~ 0	

From the supply value of 500, the demand value of 300 is satisfied. Subtract 300 from the supply value of 500 and subtract 300 from the demand value of 300. The demand of

destination 4 is fully satisfied. Hence, delete the column 4; as a result we get, the table as shown in Table 6.17.

Table 6.17: Exhausted Column 4 Deleted

Source	Destination 1	2	3	Supply
1	4	2 250	7	250
2	3	7	5	450
3	9	4	3	200
Demand	200	400 150	300	

Now, again take the minimum cost value available in the existing table and allocate it with a value of 250 in the cell (1,2).

The reduced matrix is shown in Table 6.18

Table 6.18: Exhausted Row 1 Deleted

Source	Destination 1	2	3	Supply
2	3 200	7	5	450 250

Source3	9	4	3	450
Demand	200	150	300	

In the reduced Table 6.18, the minimum value 3 exists in cell (2,1) and (3,3), which is a tie. If there is a tie, it is preferable to select a cell where maximum allocation can be made. In this case, the maximum allocation is 200 in both the cells. Choose a cell arbitrarily and allocate. The cell allocated in (2,1) is shown in Table 6.18. The reduced matrix is shown in Table 6.19.

Table 6.19: Reduced Matrix

		Destination		
		2	3	Supply
	2	7	5	250
Source	3	4	3 200	200 0
Demand		150	300 100	

Now, deleting the exhausted demand row 3, we get the matrix as shown in Table 6.20

Table 6.20: Exhausted Row 3 Deleted

Destination

2 **3** Supply

Source	**2**	7 150	5 100	0 25 / 0
Demand		150	100	
		0	0	

The initial basic feasible solution using least cost method is shown in a single Table 6.21

Table 6.21: Initial Basic Feasible Solution Using LCM Method

		Destination				
		1	**2**	**3**	**4**	**Supply**
Source	**1**	4	2 250	7	3	250
	2	3 200	7 150	5 100	8	450
	3	9	4	3 200	1 300	500
Demand		200	400	300	300 0	

Transportation Cost = (2 × 250)+ (3 × 200) + (7 × 150) + (5 × 100)+ (3 × 200) + (1 × 300)

Table 6.22: Penalty Calculation for each Row and Column

		Destination					
		1	2	3	4	Supply	Penalty
	1	4	2	7	3	250	(1)
Source	2	3	7	5	8	450	(2)
	3	9	4	3	1 / 300	~~500~~ 200	← (2)
Demand		200	400	300	~~300~~ 0		
					0		
		(1)	(2)	(2)	(2)		

Since the demand is satisfied for destination 4, delete column 4. Now again calculate the penalties for the remaining rows and columns.

Table 6.23: Exhausted Column 4 Deleted

		Destination				
		1	2	3	Supply	Penalty
	1	4	[illegible] / 250	[illegible]	~~250~~ 0	← (2)
Source	2	3	7	5	450	(2)
	3	9	4	3	200	(1)
Demand		200	~~400~~ 150	300		
		(1)	(2)	(2)		

= 500 + 600 + 1050 + 500 + 600 + 300 = Rs. 3550

Vogel's Approximation Method (VAM): The penalties for each row and column are calculated (steps given on pages 176-77) Choose the row/column, which has the maximum value for allocation. In this case there are five penalties, which have the maximum value 2. The cell with least cost is Row 3 and hence select cell (3,4) for allocation. The supply and demand are 500 and 300 respectively and hence allocate 300 in cell (3,4) as shown in Table 6.22

In the Table 6.24 shown, there are four maximum penalties of values which is 2. Selecting the least cost cell, (1,2) which has the least unit transportation cost 2. The cell (1, 2) is selected for allocation as shown in Table 6.23. Table 6.24 shows the reduced table after deleting row l.

In the Table 6.24 shown, there are four maximum penalties of values which is 2. Selecting the least cost cell, (1,2) which has the least unit transportation cost 2. The cell (1, 2) is selected for allocation as shown in Table 6.23. Table 6.24 shows the reduced table after deleting row l.

Table 6.24: Row 1 Deleted

Destination

Source	1	2	3	Supply	Penalty
2	3 200	7	5	450 250	(2)
3	9	4	3	200	(1)
Demand	200	150	300		
	(6)	(3)	(2)		

After deleting column 1 we get the table as shown in the Table 6.25 below.

Table 6.25: Column 1 Deleted

Destination

Source	2	3	Supply	Penalty
2	7	5	250	(2)
3	4 150	3	200 50	(1)
Demand	150 0	300		
	(3) -	(2)		

Finally we get the reduced table as shown in Table 6.26

Table 6.26: Final Reduced Table

		Destination	
		3	Supply
Source	2	5 250	250 0
	3	3 50	50 0
Demand		300 0	

The initial basic feasible solution is shown in Table 6.27.

Table 6.27: Initial Basic Feasible Solution

	W_1	W_2	W_3	W_4	Supply
F_1	140 17	5	9	65	140 (4) (4) (8) (48) (48)
F_2	50 20	10	12	210 65	260 (2) (2) (8) (45) (45)
F_3	10 15	100 0	250 5	65	360 (5) (5) (10) (50) __
F_4	13	220 1	10	65	220 (9) __ __ __ __ __
Demand	200	320	250	210	
	(2)	(1)	(4)	(0)	
	(2)	(5)	(4)	(0)	
	(2)	__	__	(0)	
	(2)	__	__	(0)	
	(3)	__	__	(0)	

Transportation cost = $(2 \times 250) + (3 \times 200) + (5 \times 250) + (4 \times 150) + (3 \times 50) + (1 \times 300)$

viii) 500 + 600 + 1250 + 600 + 150 + 300
ix) Rs. 3,400.00

26.8. Assignment problem

26.8 .1. Aims and objectives

In this lesson we would be able to learn assignment of various work activities using various methods of assignment problems. Solving both maximization and minimization problems and both bounded and unbounded solutions of assignment problem.

26.8 .2. Introduction

The basic objective of an assignment problem is to assign *n* number of resources to *n* number of activities so as to minimize the total cost or to maximize the allocation in such a way that the measure of effectiveness is optimized. The problem of total profit of assignment arises because available resources such as men, machines, etc., have varying degree of efficiency for performing different activities such as job. Therefore cost, profit or time for performing the different activities is different. Hence the problem is, how should the assignments be made so as to optimize (maximize or minimize) the given objective. The assignment model can be applied in many decision-making processes like determining optimum processing time in machine operators and jobs, effectiveness of teachers and subjects, designing of good plant layout, etc. This technique is found suitable for routing travelling salesmen to minimize the total travelling cost, or to maximize the sales.

26.8 .3 Mathematical structure of assignment problem

The structure of assignment problem of assigning operators to jobs is shown in Table 7.1.

Table 7.1: Structure of Assignment Problem

		Operator 1	2	……	j	…..	n
Job	1	t_{11}	t_{12}	...	t_{1j}	....	t_{1n}
	2	t_{21}	t_{22}	....	t_{2j}	....	t_{2n}
	.	.					
	I	t_{i1}	t_{i2}		t_{ij}		t_{in}
	.	.	.				
	N	t_{n1}	t_{n2}		t_{nj}		t_{nn}

Let n be the number of jobs and number of operators. t_{ij} be the processing time of job i taken by operator j. A few applications of assignment problem are:

x) assignment of employees to machines.

xi) assignment of operators to jobs.

xii) effectiveness of teachers and subjects.

xiii) allocation of machines for optimum utilization of space.

xiv) salesmen to different sales areas.

xv) clerks to various counters.

In all the cases, the objective is to minimize the total time and cost or otherwise maximize the sales and returns.

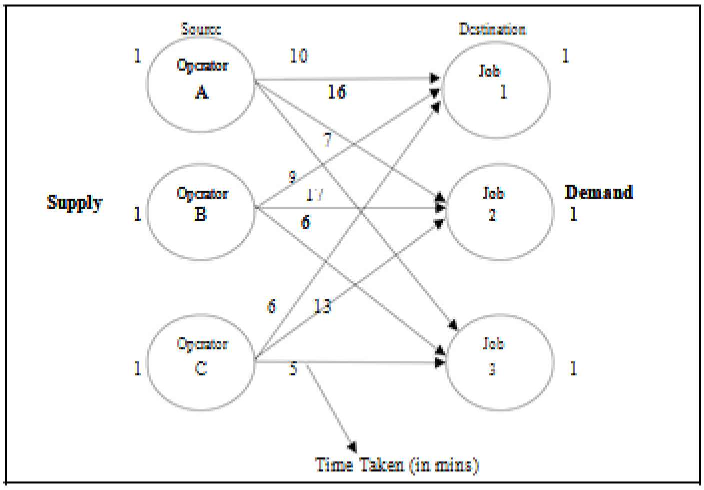
26.8 .4. Network representation of assignment problem

An assignment model is represented by a network diagram in Figure 1 for an operator – job assignment problem, given in Table 7.2 the time taken (in mins.) by operators to perform the job.

Table 7.2: Assignment Problem

Operator	Job		
	1	2	3
A	10	16	7
B	9	17	6
C	6	13	5

The assignment problem is a special case of transportation problem where all sources and demand are equal to 1



26.8 .5. Use of linear programming to solve assignment problem

A linear programming model can be used to solve the assignment problem. Consider the example shown in Table 2, to develop a linear programming model.

Let,

x11 represent the assignment of operator A to job 1 x12 represent the assignment of operator A to job 2

x13 represent the assignment of operator A to job 3

x21 represent the assignment of operator B to job 1
and so on.

Formulating the equations for the time taken by each operator,

10 x11 + 16 x12 + 7 x13 = time taken by operator A.

9 x21 + 17 x22 + 6 x23 = time taken by operator B.

6 x31 + 13 x32 + 5 x33 = time taken by operator C.

The constraint in this assignment problem is that each operator must be assigned to only one job and similarly, each job must be performed by only one operator. Taking this constraint into account, the constraint equations are as follows:

$x_{11} + x_{12} + x_{13} \leq 1$ operator A

$x_{21} + x_{22} + x_{23} \leq 1$ operator B

$x_{31} + x_{32} + x_{33} \leq 1$ operator C

$x_{11} + x_{21} + x_{31} = 1$ Job 1

$x_{12} + x_{22} + x_{32} = 1$ Job 2

$x_{13} + x_{23} + x_{33} = 1$ Job 3

Objective function: The objective function is to minimize the time taken to complete all the jobs. Using the cost data table, the following equation can be arrived at:

The objective function is,

Minimize $Z = 10 x_{11} + 16 x_{12} + 7 x_{13} + 9 x_{21} + 17 x_{22} + 6 x_{23} + 6 x_{31} + 13 x_{32} + 5 x_{33}$ The linear programming model for the problem will be,

Minimize $Z = 10 x_{11} + 16 x_{12} + 7 x_{13} + 9 x_{21} + 17 x_{22} + 6 x_{23} + 6 x_{31} + 13 x_{32} + 5 x_{33}$ subject to constraints

$x_{11} + x_{12} + x_{13} \leq 1$ ....(i)

$x_{21} + x_{22} + x_{23} \leq 1$ ....(ii)

$x_{31} + x_{32} + x_{33} \leq 1$ .... (iii)

$x_{11} + x_{12} + x_{13} = 1$ .... (iv)

$x_{12} + x_{22} + x_{32} = 1$ ....(v)

$x_{13} + x_{23} + x_{33} = 1$ .... (vi)

where, $x_{ij} \geq 0$ for i = 1,2,3 and j = 1,2,3.

26.8 .6.Types of assignment problem

The assignment problems are of two types (i) balanced and (ii) unbalanced. If the number of rows is equal to the number of columns or if the given problem is a square matrix, the problem is termed as a ***balanced assignment problem***. If the given problem is not a square matrix, the problem is termed as an ***unbalanced assignment problem***.

If the problem is an unbalanced one, add dummy rows /dummy columns as required so that the matrix becomes a square matrix or a balanced one. The cost or time values for the dummy cells are assumed as zero.

26.8.7. Hungarian method for solving assignment problem

Step 1: In a given problem, if the number of rows is not equal to the number of columns and vice versa, then add a dummy row or a dummy column. The assignment costs for dummy cells are always assigned as zero.

Step 2: Reduce the matrix by selecting the smallest element in each row and subtract with other elements in that row.

Step 3: Reduce the new matrix column-wise using the same method as given in step 2.

Step 4: Draw minimum number of lines to cover all zeros.

Step 5: If Number of lines drawn = order of matrix, then optimally is reached, so proceed to step 7. If optimally is not reached, then go to step 6.

Step 6: Select the smallest element of the whole matrix, which is **NOT COVERED** by lines. Subtract this smallest element with all other remaining elements that are **NOT COVERED** by lines and add the element at the intersection of lines. Leave the elements covered by single line as it is. Now go to step 4.

Step 7: Take any row or column which has a single zero and assign by squaring it. Strike off the remaining zeros, if any, in that row and column (X). Repeat the process until all the assignments have been made.

Step 8: Write down the assignment results and find the minimum cost/time.

Note: While assigning, if there is no single zero exists in the row or column, choose anyone zero and assign it. Strike off the remaining zeros in that column or row, and repeat the same for other assignments also. If there is no single zero allocation, it means multiple number of solutions exist. But the cost will remain the same for different sets of allocations.

Example 1: Assign the four tasks to four operators. The assigning costs are given inTable 7.4.

Table 7.4: Assignment Problem

Operators

		1	2	3	4
Tasks	A	20	28	19	13
	B	15	30	31	28
	C	40	21	20	17
	D	21	28	26	12

Solution:

Step 1: The given matrix is a square matrix and it is not necessary to add a dummy row/column

Step 2: Reduce the matrix by selecting the smallest value in each row and subtracting from other values in that corresponding row. In row A, the smallest value is 13, row B is 15, row C is 17 and row D is 12. The row wise reduced matrix is shown in Table 7.5.

Step 2: Reduce the matrix by selecting the smallest value in each row and subtracting from other values in that corresponding row. In row A, the smallest value is 13, row B is 15, row C is 17 and row D is 12. The row wise reduced matrix is shown in Table 7.5.

Table 7.5: Row-wise Reduction

Operators

		1	2	3	4
Tasks	A	7	15	6	0
	B	0	15	16	13
	C	23	4	3	0
	D	9	16	14	0

Step 3: Reduce the new matrix given in Table 6 by selecting the smallest value in each column and subtract from other values in that corresponding column. In column 1, the smallest value is 0, column 2 is 4, column 3 is 3 and column 4 is

xvi) The column-wise reduction matrix is shown in Table 7.6.

Table 7.6: Column-wise Reduction Matrix

Operators

Tasks		1	2	3	4
	A	7	11	3	6
	B	0	11	13	13
	C	23	0	0	0
	D	9	12	11	0

Step 4: Draw minimum number of lines possible to cover all the zeros in the matrix given in Table 7.7

Table 7.7: Matrix with all Zeros Covered

Operators

Tasks		1	2	3	4
	A	7	11	3	0
	B	0	11	13	13
	C	23	0	0	0
	D	9	12	11	0

No. of lines drawn = order of Matrix

The first line is drawn crossing row C covering three zeros, second line is drawn crossing column 4 covering two zeros and third line is drawn crossing column 1 (or row B) covering a single zero.

Step 5: Check whether number of lines drawn is equal to the order of the matrix, i.e., 3 ^{1} 4. Therefore optimally is not reached. Go to step 6.

Step 6: Take the smallest element of the matrix that is not covered by single line, which is 3. Subtract 3 from all other values that are not covered and add 3 at the intersection of lines. Leave the values which are covered by single line. Table 7.8 shows the details.

Table 7.8: Subtracted or Added to Uncovered Values and Intersection Lines Respectively

Operators

Tasks		1	2	3	4
	A	7	9	0	0
	B	0	9	10	13
	C	26	0	0	3
	D	9	9	8	0

Step 7: Now, draw minimum number of lines to cover all the zeros and check for optimality. Here in Table 7.9 minimum number of lines drawn is 4 which is equal to the order of matrix. Hence optimality is reached.

Table 7.9: Optimality Matrix

Operators

Tasks	1	2	3	4
A	7	9	0	0
B	0	9	10	13
C	26	0	0	3
D	9	9	8	0

No. of lines

drawn = order of matrix

Step 8: Assign the tasks to the operators. Select a row that has a single zero and assign by squaring it. Strike off remaining zeros if any in that row or column. Repeat the assignment for other tasks. The final assignment is shown in Table 7.10.

Table 7.10: Final Assignment

Operators

Tasks	**1**	**2**	**3**	**4**
A	7	9	[0]	×0
B	[0]	9	10	13
C	26	[0]	×0	3
D	9	9	8	[0]

Therefore, optimal assignment is:

Task	Operator	Cost
A	3	19
B	1	15
C	2	21
D	4	12

Total Cost = Rs. 67.00

Example 2: Solve the following assignment problem shown in Table 7.11 using Hungarian method. The matrix entries are processing time of each man in hours.

Men

Job	1	2	3	4	5
I	20	15	18	20	25
II	18	20	12	14	15
III	21	23	25	27	25
IV	17	18	21	23	20
V	18	18	16	19	20

Solution: The row-wise reductions are shown in Table 7.12

Table 7.12: Row-wise Reduction Matrix

Men

Job	1	2	3	4	5
I	5	0	3	5	10
II	6	8	0	2	3
III	0	2	4	6	4
IV	0	1	4	6	3
V	2	2	0	3	4

The column wise reductions are shown in Table 7.13.

Men

Job	1	2	3	4	5
I	5	0	3	3	7
II	6	8	0	0	0
III	0	2	4	4	1
IV	0	1	4	4	0
V	2	2	0	1	1

Matrix with minimum number of lines drawn to cover all zeros is shown in Table 7.14.

Table 7.14: Matrix will all Zeros **Covered**

Men

Job	1	2	3	4	5
I	5	0	3	3	7
II	6	8	0	0	0
III	0	2	4	4	1
IV	0	1	4	4	0
V	2	2	0	1	1

The number of lines drawn is 5, which is equal to the order of matrix. Hence optimality is reached. The optimal assignments are shown in Table 7.15.

Table 7.15: Optimal **Assignment**

Men

Job	1	2	3	4	5
I	5	[0]	3	3	7
II	6	8	0	[0]	0
III	[0]	2	4	4	1
IV	0	1	4	4	[0]
V	2	2	[0]	1	1

Therefore, the optimal solution is:

Job	Men	Time
I	2	15
II	4	14
III	1	21
IV	5	20
V	3	16

Total time = 86 hours

26.8.8. Unbalanced assignment problem

If the given matrix is not a square matrix, the assignment problem is called an **unbalanced problem.** In such type of problems, add dummy row(s) or column(s) with the cost elements as zero to convert the matrix as a square matrix. Then the assignment problem is solved by the Hungarian method.

Example 3: A company has five machines that are used for four jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following Table 7.16.

Table 7.16: Assignment Problem

Machines

		A	B	C	D	E
	1	5	7	11	6	7
	2	8	5	5	6	5
Job	**3**	6	7	10	7	3
	4	10	4	8	2	4

Solution: Convert the 4 × 5 matrix into a square matrix by adding a dummy row D5.

Table 7.17: Dummy Row D5 Added

Machines

		A	B	C	D	E
	1	5	7	11	6	7
	2	8	5	5	6	5
Job	3	6	7	10	7	3
	4	10	4	8	2	4
	D_5	0	0	0	0	0

Table 7.18: Row-wise Reduction of the Matrix

Machines

		A	B	C	D	E
	1	0	2	6	1	2
	2	3	0	0	1	0
Job	3	3	4	7	4	0
	4	8	2	6	2	0
	D_5	0	0	0	0	0

Column-wise reduction is not necessary since all columns contain a single zero. Now, draw minimum number of lines to cover all the zeros, as shown in Table 7.19.

Table 7.19: All Zeros in the Matrix Covered

Machines

Job		A	B	C	D	E
	1	0	2	6	1	2
	2	3	0	0	1	0
	3	3	4	7	4	0
	4	8	2	6	2	0
	D_5	0	0	0	0	0

Number of lines drawn [1] Order of matrix. Hence not optimal. Select the least uncovered element, i.e., 1, subtract it from other uncovered elements, add to the elements at intersection of lines and leave the elements that are covered with single line unchanged as shown in Table 7.20.

Table 7.20: Subtracted or Added to Elements

Machines

Job		A	B	C	D	E
	1	0	1	5	0	2
	2	4	0	0	1	1
	3	3	3	6	3	0
	4	8	1	5	1	0
	D_5	1	0	0	0	1

Number of lines drawn [1] Order of matrix. Hence not optimal.

Table 7.21: Again Added or Subtracted 1 from Elements

Machines

Job		A	B	C	D	E
	1	0	1	5	0	3
	2	4	0	0	1	2
	3	2	2	5	2	0
	4	7	0	4	0	0
	D_5	1	0	0	0	2

Number of lines drawn = Order of matrix. Hence optimality is reached. Now assign the jobs to machines, as shown in Table 7.22.

Table 7.22: Assigning Jobs to Machines

Machines

Job		A	B	C	D	E
	1	[0]	1	5	×0	3
	2	4	[0]	×0	1	2
	3	2	2	5	2	[0]
	4	7	×0	4	[0]	×0
	D_5	1	×0	[0]	× 0	2

Hence, the optimal solution is:

Job	Machine	Cost
1	A	5
2	B	5
3	E	3
4	D	2
D5	C	0
	Total Cost	= Rs.15.00

Example 26.8 ***4:*** In a plant layout, four different machines M_1, M_2, $M_3$and $M_4$are to be erected in a machine shop. There are five vacant areas A, B, C, D and E. Because of limited space, Machine M_2 cannot be erected at area C and Machine M_4 cannot be erected at area A. The cost of erection of machines is given in the Table 7.23.

Table 7.23: Assignment Problem

Area

Machine	A	B	C	D	E
M1	4	5	9	4	5
M_2	6	4	--	4	3
M3	4	5	8	5	1
M_4	--	2	6	1	2

Find the optimal assignment plan.

Solution: As the given matrix is not balanced, add a dummy row D5with zero cost values. Assign a high cost H for (M2, C) and (M4, A). While selecting the lowest cost element neglect the high cost assigned H, as shown in Table 7.24 below.

Table 7.24: Dummy Row D5 Added

Area

Machine	A	B	C	D	E
M1	4	5	9	4	5
M_2	6	4	H	4	3
M_3	4	5	8	5	1
M_4	H	2	6	1	2
D5	0	0	0	0	0

Row-wise reduction of the matrix, is shown in Table 7.25.

Table 7.25: Matrix Reduced Row-wise

Area

Machine		A	B	C	D	E
	M1	0	1	5	0	1
	M_2	3	1	H	1	0
	M_3	3	4	7	4	0
	M_4	H	1	5	0	1
	D_5	0	0	0	0	0

Note: Column-wise reduction is not necessary, as each column has at least one single zero. Now, Drawn to Cover all Zeros

Area

Machine		A	B	C	D	E
	M1	0	1	5	0	1
	M_2	3	1	H	1	0
	M3	3	4	7	4	0
	M_4	H	1	5	0	1
	D_5	0	0	0	0	0

Number of lines drawn ¹ Order of matrix. Hence not Optimal. Select the smallest uncovered element, in this case 1. Subtract 1 from all other uncovered element and add 1 with the elements at the intersection. The element covered by single line remains unchanged. These changes are shown in Table 7.27. Now try to draw minimum number of lines to cover all the zeros.

Table 7.27: Added or Subtracted 1 from Elements

Area

Machine		A	B	C	D	E
	M_1	0	1	5	1	2
	M_2	2	0	H	1	0
	M_3	2	3	6	4	0
	M4	H	0	4	0	1
	D_5	0	0	0	1	1

Now number of lines drawn = Order of matrix, hence optimality is reached. Optimal assignment of machines to areas are shown in Table 7.28

Table 7.28: Optimal Assignment

Area

Machine		A	B	C	D	E
	M_1	[0]	1	5	1	2
	M_2	2	[0]	H	1	x0
	M_3	2	3	6	4	[0]
	M_4	H	0	4	[0]	1
	D_5	×0	×0	[0]	1	1

Hence, the optimal solution is:

Machines	Area	Erection Cost
M_1	A	4
M_2	B	4
M_3	C	1
M_4	D	1
D_5	E	0
Total Erection Cost = Rs.10.00		

26.8.9. Restricted assignment problem

In real practice, situations may arise where a particular machine cannot be assigned to an operator because he may not be skilled enough to operate it. Because of this, no assignment is made for the operator on that machine. This situation is overcome by assigning a large value, or by assigning M. This will result in no assignment made to the restricted combinations.

Example 26.8.***5:*** Five jobs are to be assigned to five men. The cost (in Rs.) of performing the jobs by each man is given in the matrix (Table 7.29). The assignment has restrictions that Job 4 cannot be performed by Man 1 and Job 3 cannot be performed by Man 4 Find the optimal assignment of job and its cost involved.

Table 7.29: Assignment Problem

Men	**1**	**2**	**3**	**4**	**5**
1	16	12	11	x	15
2	13	15	11	16	18
3	20	21	18	19	17
4	16	13	x	16	12
5	20	19	18	17	19

Solution: Assign large value to the restricted combinations or introduce 'M', seeTable 7.30.

Table 7.30: Large Value Assignment to Restricted Combinations

		Job				
		1	2	3	4	5
	1	16	12	11	M	15
	2	13	15	11	16	18
Men	3	20	21	18	19	17
	4	16	13	M	16	12
	5	20	19	18	17	19

Table 7.31: Reducing the matrix row-wise

		Job				
		1	2	3	4	5
	1	5	1	0	M	4
	2	2	4	0	5	7
Men	3	3	4	1	2	0
	4	4	1	M	4	0
	5	3	2	1	0	1

Table 7.32: Reducing the matrix column-wise

		Job				
		1	2	3	4	5

	1	3	0	0	M	4
	2	0	3	0	5	7
Men	3	1	3	1	2	0
	4	2	0	M	4	0
	5	1	1	1	0	1

Draw minimum number of lines to cover all zeros, see Table 7.33.

Table 7.

33: All Zeros **Covered**

		Job				
		1	2	3	4	5
	1	3	0	0	M	4
	2	0	3	0	5	7
Men	3	1	3	1	2	0
	4	2	0	M	4	0
	5	1	1	1	0	1

Now, number of lines drawn = Order of matrix, hence optimality is reached (Table 7.34). Allocating Jobs to Men.

Table 7.34: Job Allocation to Men

		Job				
		1	2	3	4	5
	1	3	×0	[0]	M	4
	2	[0]	3	×0	5	7
Men	3	1	3	1	2	[0]
	4	2	[0]	M	4	×0
	5	1	1	1	[0]	1

Table 7.35: Assignment Schedule and Cost

Men	Job	Cost
1	3	11
2	1	13
3	5	17
4	2	13
5	4	17

Total Cost = Rs. 71.00

As per the restriction conditions given in the problem, Man 1 and Man 4 are not assigned to Job 4 and Job 3 respectively.

MULTIPLE AND UNIQUE SOLUTIONS

For a given Job-Men assignment problem, there can be more than one optimal solution, i.e., multiple solutions can exist. Two assignment schedules that give same results are called ***Multiple optimal solutions.*** If the problem has only one solution then the solution is said to be ***Unique solution*.**

26.8.10. Maximization problem

In maximization problem, the objective is to maximize profit, revenue, etc. Such problems can be solved by converting the given maximization problem into a minimization problem.

xvii) Change the signs of all values given in the table.

xviii) Select the highest element in the entire assignment table and subtract all the elements of the table from the highest element.

Example 6: A marketing manager has five salesmen and sales districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows (Table 7.36). Find the assignment of salesmen to districts that will result in maximum sales.

Table 7.36: Maximization Problem

		District				
		A	**B**	**C**	**D**	**E**
Salesman	**1**	32	38	40	28	40
	2	40	24	28	21	36
	3	41	27	33	30	37
	4	22	38	41	36	36
	5	29	33	40	35	39

Solution: The given maximization problem is converted into minimization problem (Table 7.37) by subtracting from the highest sales value (i.e., 41) with all elements of the given table.

Table 7.37: Conversion to Minimization Problem

District

Salesman		A	B	C	D	E
	1	9	3	1	13	1
	2	1	17	13	20	5
	3	0	14	8	11	4
	4	19	3	0	5	5
	5	12	8	1	6	2

Reduce the matrix row-wise (see Table 7.38)

Table 7.38: Matrix Reduced Row-wise

District

Salesman		A	B	C	D	E
	1	8	2	0	12	0
	2	0	16	12	19	4
	3	0	14	8	11	4
	4	19	3	0	5	5
	5	11	7	0	5	1

Reduce the matrix column-wise and draw minimum number of lines to cover all the zeros in the matrix, as shown in Table 7.39.

Table 7.39: Matrix Reduced Column-wise and Zeros Covered

District

Salesman		A	B	C	D	E
	1	8	0	0	7	0
	2	0	14	12	14	4
	3	0	12	8	6	4
	4	19	1	0	0	5
	5	11	5	0	0	1

Number of lines drawn Order of matrix. Hence not optimal.

Select the least uncovered element, i.e., 4 and subtract it from other uncovered elements, add it to the elements at intersection of line and leave the elements that are covered with single line unchanged, Table 7.40.

Table 7.40: Added & Subtracted the least Uncovered Element

District

Salesman

	A	B	C	D	E
1	12	0	0	7	0
2	0	10	8	10	0
3	0	8	4	2	0
4	23	1	0	0	5
5	15	5	0	0	1

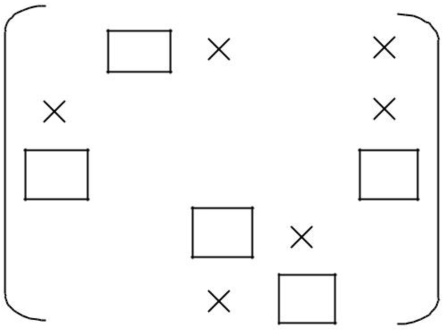
Now, number of lines drawn = Order of matrix, hence optimality is reached.

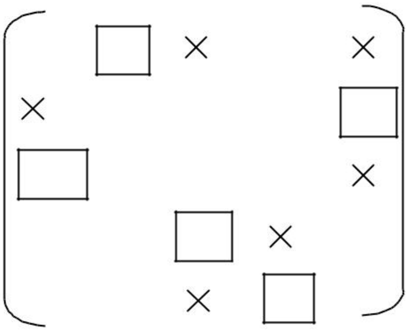
There are two alternative assignments due to presence of zero elements in cells (4, C), (4, D), (5, C) and (5, D).

Table 7.41: Two Alternative Assignments

	A	B	C	D	E
1	12		0	7	0
		0			
2	0	10	8	10	0
3	0	8	4	2	0
4	23	1	0	0	5
5	15	5	0	0	1

	A	B	C	D	E
1	12		0	7	0
1		0			
2	0	10	8	10	0
3	0	8	4	2	0
4	23	1	0	0	5
5	15	5	0	0	1





Therefore,

Assignment 1

Salesman	Districts	Sales (in '00) Rs.
1	B	38
2	A	40
3	E	37
4	C	41
5	D	35
		Total Rs. = 191.00

Assignment 2

Salesman	Districts	Sales (in '00) Rs.
1	B	38
2	E	36
3	A	41
4	C	41
5	D	35
		Total Rs. = 191.00

26.8.11. Travelling salesman problem

The 'Travelling salesman problem' is very similar to the assignment problem except that in the former, there are additional restrictions that a salesman starts from his city, visits each city once and returns to his home city, so that the total distance (cost or time) is minimum.

Procedure:

Step 1: Solve the problem as an assignment problem.

Step 2: Check for a complete cycle or alternative cycles. If the cycle is complete, Go to Step 4. If not, go to the Step 3.

Step 3: To start with, assign the next least element other than zero, (only for first allocation) and complete the assignment. Go to Step 2.

Step 4: Write the optimum assignment schedule and calculate the cost/time.

(***Note:*** If there are two non-zero values in the matrix, it means that there are two optimal solutions. Calculate the cost for the two allocations and find the optimal solution.)

Example 26.8.*7:* A Travelling salesman has to visit five cities. He wishes to start from a particular city, visit each city once and then return to his starting point. The travelling cost (in Rs.) of each city from a particular city is given below.

Table 7.42: Travelling Salesman Problem

To city

From city	A	B	C	D	E
A	á	2	5	7	1
B	6	á	3	8	2
C	8	7	á	4	7
D	12	4	6	á	5
E	1	3	2	8	á

What should be the sequence of the salesman's visit, so that the cost is minimum?

Solution: The problem is solved as an assignment problem using Hungarian method; an optimal solution is reached as shown in Table 7.43.

Table 7.43: Optimal Solution Reached Using Hungarian Method

To city

From city	A	B	C	D	E
A	á	1	3	6	[0]
B	4	á	[0]	6	×0
C	4	3	á	[0]	3
D	8	[0]	1	á	1
E	[0]	2	×0	7	á

In this assignment, it means that the travelling salesman will start from city A, then go to city E and return to city A without visiting the other cities. The cycle is not complete. To overcome this situation, the next highest element can be assigned to start with. In this case it is 1, and there are three 1's. Therefore, consider all these 1's one by one and find the route which completes the cycle.

***Case 1*:**Make the assignment for the cell (A, B) which has the value 1. Now, make the assignments for zeros in the usual manner. The resulting assignments are shown in Table 7.44.

Table 7.44: Resulting Assignment

To city

From city

	A	B	C	D	E
A	á	1	3	6	×0
B	4	á	0	6	×0
C	4	3	á	0	3
D	8	×0	1	á	1
E	0	2	×0	7	á

The assignment shown in Table 7.42 gives the route sequence

A ® B, B ® C, C ® D, D ®E and E ® A.

The travelling cost to this solution is

i) 2000 + 3000 + 4000 + 5000 + 1000

ii) Rs.15,000.00

***Case 2*:**If the assignment is made for cell (D, C) instead of (D, E), the feasible solution cannot be obtained. The route for the assignment will be A ® B ® C ® D® C. In this case, the salesman visits city C twice and cycle is not complete.

Therefore the sequence feasible for this assignment is

A ® B ® C ® D ® E ® A with the travelling cost of Rs.15,000.00

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